MATHEMATICS

ACCOUNTING PROFESSION

for Rwandan Schools

Senior 4 Teacher's Guide

FOREWORD

Dear Teachers,

Rwanda Basic Education Board is honoured to present the teacher's guide for Mathematics in the Accounting Profession Option. This book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics Subject. The Rwandan educational philosophy is to ensure that students achieve full potential at every level of education which will prepare them to be well integrated in society and exploit employment opportunities.

Specifically, the curriculum for Accounting Profession Option was reviewed to train quality Accountant Technicians who are qualified, confident and efficient for job opportunities and further studies in Higher Education in different programs under Accounting career advancement.

In line with efforts to improve the quality of education, the government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate their learning process. Many factors influence what students learn, how well they learn and the competences they acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials.

High Quality Technician Accounting program is an important component of Finance and Economic development of the Rwanda Vision 2050, ***"The Rwanda We Want"*** that aims at transforming the country's socioeconomic status. The qualified Technicians accountant will significantly play a major role in the mentioned socioeconomic transformation journey. Mathematics textbooks and teacher's guide were elaborated to provide the mathematical operations, algebraic functions and equations, and basic statistics that are necessary to train a Technician Accountant capable of successfully perform his/her duties.

The ambition to develop a knowledge-based society and the growth of regional and global competition in the jobs market has necessitated the shift to a competence-based curriculum.

The Mathematics teacher's guide provides active teaching and learning techniques that engage students to develop competences. In view of this, your role as a Mathematics teacher is to:

- Plan your lessons and prepare appropriate teaching materials.
- Organize group discussions for students considering the importance of social constructivism suggesting that learning occurs more effectively when the students works collaboratively with more knowledgeable and

experienced people.

- Engage students through active learning methods such as inquiry methods, group discussions, research, investigative activities and group or individual work activities.
- Provide supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation.
- Support and facilitate the learning process by valuing students' contributions in the class activities.
- Guide students towards the harmonization of their findings.
- Encourage individual, pair and group evaluation of the work done in the classroom and use appropriate competence-based assessment approaches and methods.

To facilitate you in your teaching activities, the content of this book is self-explanatory so that you can easily use it. It is divided in 3 parts:

The part I explains the structure of this book and gives you the methodological guidance;

The part II gives a sample lesson plan;

The part III details the teaching guidance for each concept given in the student book.

Even though this Teacher's guide contains the guidance on solutions for all activities given in the student's book, you are requested to work through each question before judging student's findings.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, UR Lecturers, Teachers from TTC and General Education and experts from different Education partners for their technical support. A word of gratitude goes also to the administration of Universities, Head Teachers and TTCs principals who availed their staff for various activities.

Dr. MBARUSHIMANA Nelson
Director General, REB.

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I wish to express my appreciation to the people who played a major role in the development of this teacher`s guide for Mathematics in the Accounting profession option. It would not have been successful without active participation of different education stakeholders.

I owe gratitude to different universities and schools in Rwanda that allowed their staff to work with REB in the in-house textbooks production initiative.

I wish to extend my sincere gratitude to lecturers and teachers whose efforts during writing exercise of this teacher`s guide was very much valuable.

Finally, my word of gratitude goes to the Rwanda Basic Education Board staffs who were involved in the whole process of in-house textbook writing.

MURUNGI Joan

Head of Curriculum, Teaching and Learning Resources Department / REB

TABLE OF CONTENT

PART I. GENERAL INTRODUCTION

1.1 The structure of the guide

The teacher's guide of Mathematics is composed of three parts:

The Part I concerns general introduction that discusses methodological guidance on how best to teach and learn Mathematics, developing competences in teaching and learning, addressing cross-cutting issues in teaching and learning and Guidance on assessment.

Part II presents a sample lesson plan. This lesson plan serves to guide the teacher on how to prepare a lesson in Mathematics.

The Part III is about the structure of a unit and the structure of a lesson. This includes information related to the different components of the unit and these components are the same for all units. This part provides information and guidelines on how to facilitate students while working on learning activities. More other, all application activities from the textbook have answers in this part.

1.2 Methodological guidance

1.2.1 Developing competences

Since 2015 Rwanda shifted from a knowledge based to a competence-based curriculum for pre-primary, primary, secondary education and recently the curriculum for profession options such as TTC, Associate Nurse and Accounting programs. This called for changing the way of learning by shifting from teacher centred to a learner centred approach. Teachers are not only responsible for knowledge transfer but also for fostering students' learning achievement and creating safe and supportive learning environment. It implies also that students have to demonstrate what they are able to transfer the acquired knowledge, skills, values and attitude to new situations.

The competence-based curriculum employs an approach of teaching and learning based on discrete skills rather than dwelling on only knowledge or the cognitive domain of learning. It focuses on what learner can do rather than what learner knows. Students develop competences through subject unit with specific learning objectives broken down into knowledge, skills and attitudes/ values through learning activities.

In addition to the competences related to Mathematics, students also develop generic competences which should promote the development of the higher order thinking skills and professional skills in Mathematics teaching. Generic competences are developed throughout all units of Mathematics as follows:

Generic competences	**Ways of developing generic competences**
Critical thinking	All activities that require students to calculate, convert, interpret, analyse, compare and contrast, etc have a common factor of developing critical thinking into students
Creativity and innovation	All activities that require students to plot a graph of a given algebraic data, to organize and interpret statistical data collected and to apply skills in solving problems of production/ finance/ economic have a common character of developing creativity into students
Research and problem solving	All activities that require students to make a research and apply their knowledge to solve problems from the real-life situation have a character of developing research and problem solving into students.
Communication	During Mathematics class, all activities that require students to discuss either in groups or in the whole class, present findings, debate …have a common character of developing communication skills into students.
Co-operation, interpersonal relations and life skills	All activities that require students to work in pairs or in groups have character of developing cooperation and life skills among students.
Lifelong learning	All activities that are connected with research have a common character of developing into students a curiosity of applying the knowledge learnt in a range of situations. The purpose of such kind of activities is for enabling students to become life-long students who can adapt to the fast-changing world and the uncertain future by taking initiative to update knowledge and skills with minimum external support.
Professional skills	Specific instructional activities and procedures that a teacher may use in the class room to facilitate, directly or indirectly, students to be engaged in learning activities. These include a range of teaching skills: the skill of questioning, reinforcement, probing, explaining, stimulus variation, introducing a lesson; illustrating with examples, using blackboard, silence and non-verbal cues, using audio – visual aids, recognizing attending behaviour and the skill of achieving closure.

The generic competences help students deepen their understanding of Mathematics and apply their knowledge in a range of situations. As students develop generic competences they also acquire the set of skills that employers look for in their employees, and so the generic competences prepare students for the world of work.

1.2.2 Addressing cross cutting issues

Among the changes brought by the competence-based curriculum is the integration of cross cutting issues as an integral part of the teaching learning process-as they relate to and must be considered within all subjects to be appropriately addressed. The eight cross cutting issues identified in the national curriculum framework are: *Comprehensive Sexuality Education, Environment and Sustainability, Financial Education, Genocide studies, Gender, Inclusive Education, Peace and Values Education, and Standardization Culture*.

Some cross-cutting issues may seem specific to particular learning areas/subjects but the teacher need to address all of them whenever an opportunity arises. In addition, students should always be given an opportunity during the learning process to address these cross-cutting issues both within and out of the classroom.

Below are examples of how crosscutting issues can be addressed:

Cross-Cutting Issue	Ways of addressing cross-cutting issues
Comprehensive Sexuality Education: The primary goal of introducing Comprehensive Sexuality Education program in schools is to equip children, adolescents, and young people with knowledge, skills and values in an age appropriate and culturally gender sensitive manner so as to enable them to make responsible choices about their sexual and social relationships, explain and clarify feelings, values and attitudes, and promote and sustain risk reducing behaviour.	Using different charts and their interpretation, Mathematics teacher should lead students to discuss the following situations: "Alcohol abuse and unwanted pregnancies" and advise students on how they can fight against them. Some examples can be given when learning statistics, powers, logarithms and the related graphical interpretation.

Environment and Sustainability: Integration of Environment, Climate Change and Sustainability in the curriculum focuses on and advocates for the need to balance economic growth, society well-being and ecological systems. Students need basic knowledge from the natural sciences, social sciences, and humanities to understand to interpret principles of sustainability.	Using Real life models or students' experience, Mathematics Teachers should lead students to illustrate the situation of "population growth" and discuss its effects on the environment and sustainability.
Financial Education: The integration of Financial Education into the curriculum is aimed at a comprehensive Financial Education program as a precondition for achieving financial inclusion targets and improving the financial capability of Rwandans so that they can make appropriate financial decisions that best fit the circumstances of one's life.	Through different examples and calculations on interest (simple and compound interests), interest rate problems, total revenue functions and total cost functions, supply and demand functions Mathematics Teachers can lead students to discuss how to make appropriate financial decisions.
Gender: At school, gender will be understood as family complementarities, gender roles and responsibilities, the need for gender equality and equity, gender stereotypes, gender sensitivity, etc.	Mathematics Teachers should address gender as cross-cutting issue through assigning leading roles in the management of groups to both girls and boys and providing equal opportunity in the lesson participation and avoid any gender stereotype in the whole teaching and learning process.

Inclusive Education: Inclusion is based on the right of all students to a quality and equitable education that meets their basic learning needs and understands the diversity of backgrounds and abilities as a learning opportunity.	Firstly, Mathematics Teachers need to identify/recognize students with special needs. Then by using adapted teaching and learning resources while conducting a lesson and setting appropriate tasks to the level of students, they can cater for students with special education needs. They must create opportunity where students can discuss how to cater for students with special educational needs.
Peace and Values Education: Peace and Values Education (PVE) is defined as education that promotes social cohesion, positive values, including pluralism and personal responsibility, empathy, critical thinking and action in order to build a more peaceful society.	Through a given lesson, a teacher should: Set a learning objective which is addressing positive attitudes and values, Encourage students to develop the culture of tolerance during discussion and to be able to instil it in colleagues and cohabitants; Encourage students to respect ideas from others.
Standardization Culture: Standardization Culture in Rwanda will be promoted through formal education and plays a vital role in terms of health improvement, economic growth, industrialization, trade and general welfare of the people through the effective implementation of Standardization, Quality Assurance, Metrology and Testing.	With different word problems related to the effective implementation of Standardization, Quality Assurance, Metrology and Testing, students can be motivated to be aware of health improvement, economic growth, industrialization, trade and general welfare of the people.

1.2.3 Guidance on how to help students with special education needs in classroom

In the classroom, students learn in different way depending to their learning pace, needs or any other special problem they might have. However, the teacher has the responsibility to know how to adopt his/her methodologies and approaches in order to meet the learning need of each student in the classroom. Also teachers need to understand that student with special needs, need to be taught differently or need some accommodations to enhance the learning environment. This will be done depending to the subject and the nature of the lesson.

In order to create a well-rounded learning atmosphere, teachers need to:

- Remember that students learn in different ways so they have to offer a variety of activities (e.g. role-play, music and singing, word games and quizzes, and outdoor activities);
- Maintain an organized classroom and limits distraction. This will help students with special needs to stay on track during lesson and follow instruction easily;
- Vary the pace of teaching to meet the needs of each student. Some students process information and learn more slowly than others;
- Break down instructions into smaller, manageable tasks. Students with special needs often have difficulty understanding long-winded or several instructions at once. It is better to use simple, concrete sentences in order to facilitate them understand what you are asking.
- Use clear consistent language to explain the meaning (and demonstrate or show pictures) if you introduce new words or concepts;
- Make full use of facial expressions, gestures and body language;
- Pair a student who has a disability with a friend. Let them do things together and learn from each other. Make sure the friend is not over protective and does not do everything for the one with disability. Both students will benefit from this strategy;
- Use multi-sensory strategies. As all students learn in different ways, it is important to make every lesson as multi-sensory as possible. Students with learning disabilities might have difficulty in one area, while they might excel in another. For example, use both visual and auditory cues.
- Below are general strategies related to each main category of disabilities and how to deal with every situation that may arise in the classroom. However, the list is not exhaustive because each student is unique with different needs and that should be handled differently.

Strategy to help students with developmental impairment:

- Use simple words and sentences when giving instructions;
- Use real objects that students can feel and handle. Rather than just working abstractly with pen and paper;
- Break a task down into small steps or learning objectives. The student should start with an activity that she/he can do already before moving on to something that is more difficult;
- Gradually give the student less help;
- Let the student with disability work in the same group with those without disability.

Strategy to help students with visual impairment:

- Help students to use their other senses (hearing, touch, smell and taste) and carry out activities that will promote their learning and development;
- Use simple, clear and consistent language;
- Use tactile objects to help explain a concept;
- If the student has some sight, ask him/her what he/she can see;
- Make sure the student has a group of friends who are helpful and who allow him/her to be as independent as possible;
- Plan activities so that students work in pairs or groups whenever possible;

Strategy to help students with hearing disabilities or communication difficulties

- Always get the student's attention before you begin to speak;
- Encourage the student to look at your face;
- Use gestures, body language and facial expressions;
- Use pictures and objects as much as possible.
- Keep background noise to a minimum.

Strategies to help students with physical disabilities or mobility difficulties:

- Adapt activities so that students who use wheelchairs or other mobility aids, can participate.
- Ask parents/caregivers to assist with adapting furniture e.g. the height of a table may need to be changed to make it easier for a student to reach it or fit their legs or wheelchair under;
- Encourage peer support when needed;
- Get advice from parents or a health professional about assistive devices if the student has one.

Adaptation of assessment strategies:

At the end of each unit, the teacher is advised to provide additional activities to help students achieve the key unit competence. These assessment activities are for remedial, consolidation and extension designed to cater for the needs of all categories of students; slow, average and gifted students respectively. Therefore, the teacher is expected to do assessment that fits individual students.

Remedial activities	After evaluation, slow students are provided with lower order thinking activities related to the concepts learnt to facilitate them in their learning. These activities can also be given to assist deepening knowledge acquired through the learning activities for slow students.
Consolidation activities	After introduction of any concept, a range number of activities can be provided to all students to enhance/ reinforce learning.
Extended activities	After evaluation, gifted and talented students can be provided with high order thinking activities related to the concepts learnt to make them think deeply and critically. These activities can be assigned to gifted and talented students to keep them working while other students are getting up to required level of knowledge through the learning activity.

1.2.4. Guidance on assessment

Assessment is an integral part of teaching and learning process. The main purpose of assessment is for improvement of learning outcomes. Assessment for learning/ Continuous/ formative assessment intends to improve students' learning and teacher's teaching whereas assessment of learning/summative assessment intends to improve the entire school's performance and education system in general.

Continuous/ formative assessment

It is an on-going process that arises during the teaching and learning process. It includes lesson evaluation and end of sub unit assessment. This formative assessment should play a big role in teaching and learning process. The teacher should encourage individual, pair and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods.

Formative assessment is used to:

- Determine the extent to which learning objectives are being achieved and competences are being acquired and to identify which students need remedial interventions, reinforcement as well as extended activities. The application activities are developed in the student
- book and they are designed to be given as remedial, reinforcement, end lesson assessment, homework or assignment
- Motivate students to learn and succeed by encouraging students to read, or learn more, revise, etc.
- Check effectiveness of teaching methods in terms of variety, appropriateness, relevance, or need for new approaches and strategies. Mathematics teachers need to consider various aspects of the instructional process including appropriate language levels, meaningful examples, suitable methods and teaching aids/ materials, etc.
- Help students to take control of their own learning.

In teaching Mathematics, formative or continuous assessment should compare performance against instructional objectives. Formative assessment should measure the student's ability with respect to a criterion or standard. For this reason, it is used to determine what students can do, rather than how much they know.

Summative assessment

The assessment can serve as summative and informative depending to its purpose. The end unit assessment will be considered summative when it is done at end of unit and want to start a new one.

It will be formative assessment, when it is done in order to give information on the progress of students and from there decide what adjustments need to be done.

The assessment done at the end of the term, end of year, is considered as summative assessment so that the teacher, school and parents are informed of the achievement of educational objective and think of improvement strategies. There is also end of level/ cycle assessment in form of national examinations.

When carrying out assessment?

Assessment should be clearly visible in lesson, unit, term and yearly plans.

- Before learning (diagnostic): At the beginning of a new unit or a section of work; assessment can be organized to find out what students already know / can do, and to check whether the students are at the same level.

- During learning (formative/continuous): When students appear to be having difficulty with some of the work, by using on-going assessment (continuous). The assessment aims at giving students support and feedback.
- After learning (summative): At the end of a section of work or a learning unit, the Mathematics Teacher has to assess after the learning. This is also known as Assessment of Learning to establish and record overall progress of students towards full achievement. Summative assessment in Rwandan schools mainly takes the form of written tests at the end of a learning unit or end of the month, and examinations at the end of a term, school year or cycle.

Instruments used in assessment.

- **Observation:** This is where the Mathematics teacher gathers information by watching students interacting, conversing, working, playing, etc. A teacher can use observations to collect data on behaviours that are difficult to assess by other methods such as attitudes, values, and generic competences and intellectual skills. It is very important because it is used before the lesson begins and throughout the lesson since the teacher has to continue observing each and every activity.
- **Questioning**
  a) Oral questioning: a process which requires a student to respond verbally to questions
  b) Class activities/ exercise: tasks that are given during the learning/ teaching process
  c) Short and informal questions usually asked during a lesson
  d) Homework and assignments: tasks assigned to students by their teachers to be completed outside of class.

Homework assignments, portfolio, project work, interview, debate, science fair, Mathematics projects and Mathematics competitions are also the different forms/ instruments of assessment.

1.2.5. Teaching methods and techniques that promote active learning

The different learning styles for students can be catered for, if the teacher uses active learning whereby students are really engaged in the learning process.

The main teaching methods used in Mathematics are the following:

- **Dogmatic method** (the teacher tells the students what to do, What to observe, How to attempt, How to conclude)
- **Inductive-deductive method:** Inductive method is to move from specific examples to generalization and deductive method is to move from

generalization to specific examples.

- **Analytic-synthetic method:** Analytic method proceeds from unknown to known, 'Analysis' means 'breaking up' of the problem in hand so that it ultimately gets connected with something obvious or already known. Synthetic method is the opposite of the analytic method. Here one proceeds from known to unknown.
- **Skills lab method:** Skills lab method is based on the maxim "learning by doing." It is a procedure for stimulating the activities of the students and to encourage them to make discoveries through practical activities.
- **Problem solving method, Project method and Seminar Method.**

The following are some active techniques to be used in Mathematics:

- Group work
- Research
- Probing questions
- Practical activities (drawing, plotting, interpreting graphs)
- Modelling
- Brainstorming
- Quiz Technique
- Discussion Technique
- Scenario building Technique

What is Active learning?

Active learning is a pedagogical approach that engages students in doing things and thinking about the things they are doing. Students play the key role in the active learning process. They are not empty vessels to fill but people with ideas, capacity and skills to build on for effective learning. Thus, in active learning, students are encouraged to bring their own experience and knowledge into the learning process.

The role of the teacher in active learning	The role of students in active learning
– The teacher engages students through active learning methods such as inquiry methods, group discussions, research, investigative activities, group and individual work activities. – He/she encourages individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods. – He provides supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation. – Teacher supports and facilitates the learning process by valuing students' contributions in the class activities.	A learner engaged in active learning: – Communicates and shares relevant information with fellow students through presentations, discussions, group work and other learner-centred activities (role play, case studies, project work, research and investigation); – Actively participates and takes responsibility for his/her own learning; – Develops knowledge and skills in active ways; – Carries out research/investigation by consulting print/online documents and resourceful people, and presents their findings; – Ensures the effective contribution of each group member in assigned tasks through clear explanation and arguments, critical thinking, responsibility and confidence in public speaking – Draws conclusions based on the findings from the learning activities.

Main steps for a lesson in active learning approach

All the principles and characteristics of the active learning process highlighted above are reflected in steps of a lesson as displayed below. Generally, the lesson is divided into three main parts whereby each one is divided into smaller steps to make sure that students are involved in the learning process. Below are those main part and their small steps:

1) Introduction

Introduction is a part where the teacher makes connection between the current and previous lesson through appropriate technique. The teacher opens short discussions to encourage students to think about the previous learning experience and connect it with the current instructional objective. The teacher reviews the prior knowledge, skills and attitudes which have a link with the new concepts to create good foundation and logical sequencings.

2) Development of the new lesson

The development of a lesson that introduces a new concept will go through the following small steps: discovery activities, presentation of students' findings, exploitation, synthesis/summary and exercises/application activities.

- **Discovery activity**

Step 1:

- The teacher discusses convincingly with students to take responsibility of their learning
- He/she distributes the task/activity and gives instructions related to the tasks (working in groups, pairs, or individual to prompt / instigate collaborative learning, to discover knowledge to be learned)

Step 2:

- The teacher let students work collaboratively on the task;
- During this period the teacher refrains to intervene directly on the knowledge;
- He/she then monitors how the students are progressing towards the knowledge to be learned and boosts those who are still behind (but without communicating to them the knowledge).

- **Presentation of students' findings/productions**
  - In this part, the teacher invites representatives of groups to present their productions/findings.
  - After three/four or an acceptable number of presentations, the teacher decides to engage the class into exploitation of students' productions.
- **Exploitation of students' findings/ productions**
  - The teacher asks students to evaluate the productions: which ones are correct, incomplete or false
  - Then the teacher judges the logic of the students' products, corrects those which are false, completes those which are incomplete, and confirms those which are correct.
- **Institutionalization or harmonization (summary/conclusion/ and examples)**
  - The teacher summarizes the learned knowledge and gives examples which illustrate the learned content.
- **Application activities**

- Exercises of applying processes and products/objects related to learned unit/sub-unit
- Exercises in real life contexts
- Teacher guides students to make the connection of what they learnt to real life situations.
- At this level, the role of teacher is to monitor the fixation of process and product/object being learned.

3) Assessment

In this step the teacher asks some questions to assess achievement of instructional objective. During assessment activity, students work individually on the task/activity. The teacher avoids intervening directly. In fact, results from this assessment inform the teacher on next steps for the whole class and individuals. In some cases, the teacher can end with a homework/ assignment. Doing this will allow students to relay their understanding on the concepts covered that day. Teacher leads them not to wait until the last minute for doing the homework as this often results in an incomplete homework set and/or an incomplete understanding of the concept.

PART II: SAMPLE LESSON

School:……………………………………….. Teacher's Names:………………………………..

Term	Date	Subject	Class	Unit Nº	Lesson Nº	Duration	Class size
------	----/------/-----	MATHEMATICS	S4	1	7 of 11	40minutes	40 student-teachers

Type of Special Educational Needs to be catered for in this lesson and number of students in each category	2 students with hearing impairment will be seat near the teacher and the use of gestures will be improved in the lesson.
Unit title	Basic concepts of algebra
Key Unit Competence:	Apply algebraic principles and concepts in solving production, financial and economical related problems
Title of the lesson	Indices/powers/exponents
Instructional Objective	Through the given activities written on flash cards, students should be able to solve finance/ production / economic problems involving powers accurately using properties of powers.
Plan for this Class (location: in / outside)	Inside the classroom
Learning Materials (for all students)	Flash Cards, papers, Pens, Exercise Books, other supporting teaching aids such as Chalks and Chalkboard, etc...
References	S4 Student's book and Teacher's guide of Mathematics.

<table>
<tr><td rowspan="3">Steps and Timing</td><td colspan="2">Description of teaching and learning activities</td><td rowspan="3">Competences and Cross-Cutting Issues to be addressed</td></tr>
<tr><td colspan="2">Students are organized into small groups and provided by clear instructions to discuss and work out the activity 1.2.1. The group members present their findings and teacher facilitates students to capture the key concepts of the lesson through harmonization.

Teacher use various probing questions to guide students to explore examples and content related to indices/exponent/powers

Finally, the Students are assigned to individual tasks / application activity 1.2.1 and the collective correction is done on the chalk board.</td></tr>
<tr><td>Teachers activities</td><td>Students activities</td></tr>
<tr><td>Introduction

5 min
▪ Discovery activity (Question 1)</td><td>Indices/powers/exponents
– teacher distributes flash cards to students in their small group discussions and invite them to brainstorm on the activity 1.2.1, question 1;
– teacher moves around to help those who are struggling and guides them in finding definitions and properties of powers.</td><td>– Students receive flashcards, discuss and brainstorm on the activity 1.2.1, question 1
– They guess the definition of power and explore properties of powers.</td><td>– Cooperation is addressed through group work where team working spirit is developed while students are working together and in small group discussions.</td></tr>
</table>

	– Teacher invites students to present their findings. – teacher harmonizes the answers from presentation.	– Group representatives present findings from groups and other students participate actively in the presentation by providing comments or by asking questions.	– Communication skills are developed through group discussions and presentation of findings.
Development of the lesson: 25min ▪ **Discovery activity (Question 2)**	– Teacher gives instructions, invites students to brainstorm in their small groups the activity 1.2.1, question 2, examples and properties of powers. in the student's book. – Teacher moves around to each group, ask probing questions in order to help struggling students. – Teacher invites students to present their findings.	– In their respective groups, Students discuss and brainstorm on activity 1.2.1, question 2, examples and properties of powers. – Students present their findings.	– Critical thinking, problem solving skills are developed through analysing and solving Mathematical problems that involve powers and their properties. – Cooperation and communication skills are developed during presentations and group discussions. – Inclusive education is addressed by providing the remediation activities and tasks to struggling students.

▪ **Presentation of findings**	– Teacher harmonize the students' findings and help them to summarize the learned knowledge and give examples which illustrate the properties of powers.	– Guided by the teacher, students summarize the lesson by highlighting the properties of powers, and take notes.	
▪ **Application activities**	– Teacher asks students to individually work out the application activity 1.2.1, question 1 (sub-questions a, b and c) and then request them to do a collective correction on the chalk board	– Individually, Students work out the application activity 1.2.1, question 1 (sub-questions a, b and c) in their textbook.	
Conclusion 10 min ▪ **Assessment**	Teacher asks students to individually work out the application activity 1.2.1, question 1 (sub-questions d and e).	Students work out independently the application activity 1.2.1, question 1 (sub-questions d and e).	**Critical thinking** is developed through analysing and solving Mathematical problem that involve powers.

▪ **Homework**	Teacher gives the homework to students.	Individually, students work out the application activity 1.2.1, question 2 by giving examples where powers are used in real life.	P**roblem solving skills** is developed through giving real life examples of application of powers. **Financial Education** is developed while students are connecting powers with the real life problems related to production, finance and economics.
Teacher self-evaluation	To be completed after receiving the feed-back from the Students.		

PART III: UNIT DEVELOPMENT

UNIT 1

BASIC CONCEPTS OF ALGEBRA

1.1 Key unit competence

Apply algebraic principles and concepts in solving production, financial and economical related problems

1.2 Prerequisite

The students will perform well in this unit if they have a good background on

- Sets and operations on sets learnt in Ordinary Level (Senior 1);
- Numerical calculations;
- Sets representations (Venn diagrams).

1.3 Cross-cutting issues to be addressed

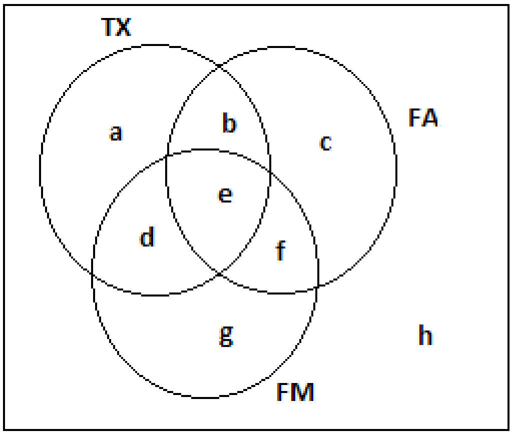
- Inclusive education (promoting education for all in teaching and learning process)
- Peace and value Education (respect the views and thoughts of others during class discussions)
- Gender: Provide equal opportunity for boys and girls to participate in class

1.4 Guidance on introductory activity 1

- Invite students to work in group, discuss and find out the answers for the introductory activity 1 from the student's book
- Facilitate students' discussions and ask them to avoid noise or other unnecessary conversations.
- During discussions, let students think of different ways to solve the given problem
- Walk around in all groups to provide assistance if necessary.
- Invite group members to present their findings and encourage boys and girls to actively participate in presentations.

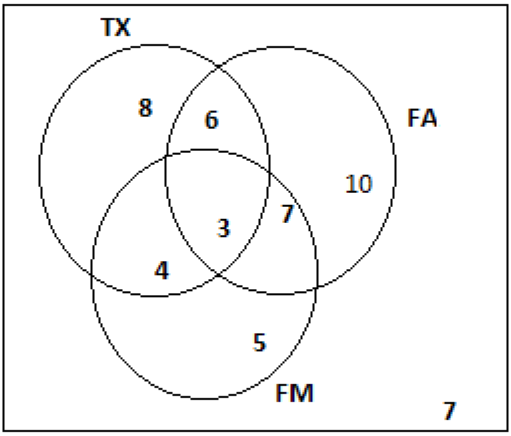
– Basing on students' experience, prior knowledge and abilities shown in answering the questions for this introductory activity 1, use different probing questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answer for introductory activity 1



From the above Venn diagram, from the given data there are 3 students in region e and 7 in region h. since 7 students taking taxation and financial management, we have n(d)+n(e)=7.

Where d=7-3=4, since 10 students taking Financial management include region e and f there must be 10-3=7 in region f. Since 9 students taking taxation and Financial Accounting there must be 9-3=6 in region b. Now, 21 students taking taxation. this include students from region a, b, d and e. students in region a are given by 21-4-3=8. Hence students taking only taxation are 8.



1.5. List of lessons

Headings	#	Lesson title/ sub-headings	Learning objectives	Number of periods
1.1. Set theory and its applications	0	Introductory activity 1	Arouse the curiosity of students on the content of unit 1.	1
	1	Sets of numbers	Classify numbers into naturals (counting and whole numbers), integers, decimals, rational, irrationals, and real numbers.	1
	2	Operations and properties on numbers	Carry out mathematical operations on numbers	2
	3	Percentages and ratios	Apply percentages and ratios to solve financial, production and economical related problems	1
	4	Venn diagrams and operations on sets	Use Venn diagram in solving production, financial and economical related problems	4
	5	Applications of set theory in financial, production and economical related problems	Apply set operations in solving production, financial and economical related problems	2

1.2. Indices/ powers/ exponents, Surds and absolute value, decimal and Naperean logarithms, and applications	1	Indices/powers/ exponents	Solve mathematical and financial related problems	1
	2	Surds and absolute value	Appreciate the importance of surds and absolute value in solving financial related problems	2
	3	Decimal and Naperian logarithms	Transform a logarithmic expression to equivalent power or radical form and vice versa.	1
	4	Applications of exponents and logarithms	Use powers and decimal logarithm to optimize simple problems related to production, finance and economics.	2
1.3 End unit assessment				1

Lesson 1: Sets of numbers

a) Learning objective:

Classify numbers into naturals (counting and whole numbers), integers, decimals, rational, irrationals, and real numbers.

b) Teaching resources:

Student's book and other reference textbooks, Mathematical set, calculator, Manila paper, graph paper, ruler, markers, pens, pencils, different graphs on charts etc.

c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they are skilled enough in the content of sets learnt in ordinary level (Senior 1).

d) Learning activities

- Invite Students to work in group and do the activity 1.1.1 form the S4 Mathematics student's book;
- Move around for facilitating students where necessary and give more

clarification on eventual challenges they may face during their work;

– Ask randomly some groups to present their findings to the whole class;
– As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills on set of numbers and their representations.
– Ask students to do the application activity 1.1.1. and evaluate whether lesson objectives were achieved to assess their competences.

Note that recurring or repeating decimal numbers are classified in the set of rational numbers

Answers for Activity 1.1.1

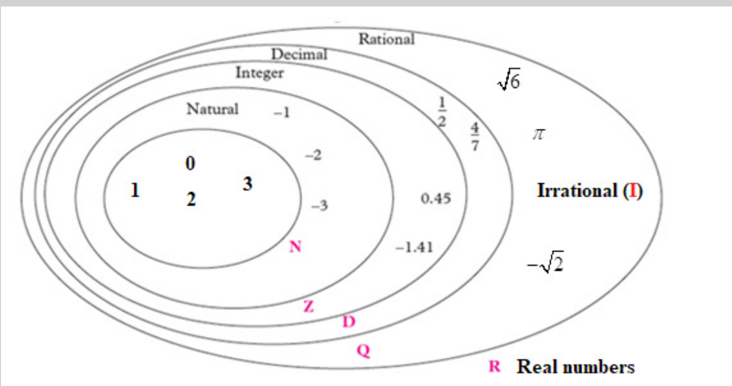
1. Lead students to list sets that they already know, their lists may include: $\mathbb{N}, \mathbb{Z}, \mathrm{D}, \mathbb{Q}, \mathbb{R}, \ldots$.The numbers we are using in counting related to all positive numbers, are called Natural numbers denoted by $\mathbb{N}$; integers are whole numbers which are either negative or positive and includes zero. The set of integers is represented by $\mathbb{Z}$; the set of limited decimal is D, the set of rational numbers represented by $\mathbb{Q}$ and the set of irrational numbers I. The set of real numbers is denoted by $\mathbb{R}$ and it includes the set of rational numbers and the set of irrational numbers.

2. Examples: $\{1,2,3\} \subset \mathbb{N}$, $\{-3,-2,-1,1,2,3\} \subset \mathbb{Z}$, $\left\{\frac{1}{2}, \frac{4}{7}\right\} \subset \mathbb{Q}$,

   $\left\{-3, 0.45, \frac{1}{2}, 2, \sqrt{3}\right\} \subset \mathbb{R}$

3. In fact, their relationship is that $\mathbb{N} \subset \mathbb{Z} \subset D \subset \mathbb{Q} \subset \mathbb{R}$.

   They can be summarized in the following figure with examples of numbers in each set:



e) Answers for the application activity 1.1.1

1. Integers lying between -7 and -3 are given in the following Roster form $\{-6,-5,-4\}$

   Integers lying between -7 and -3 are given in the following builder form $A=\{x:-7<x<-3\}$

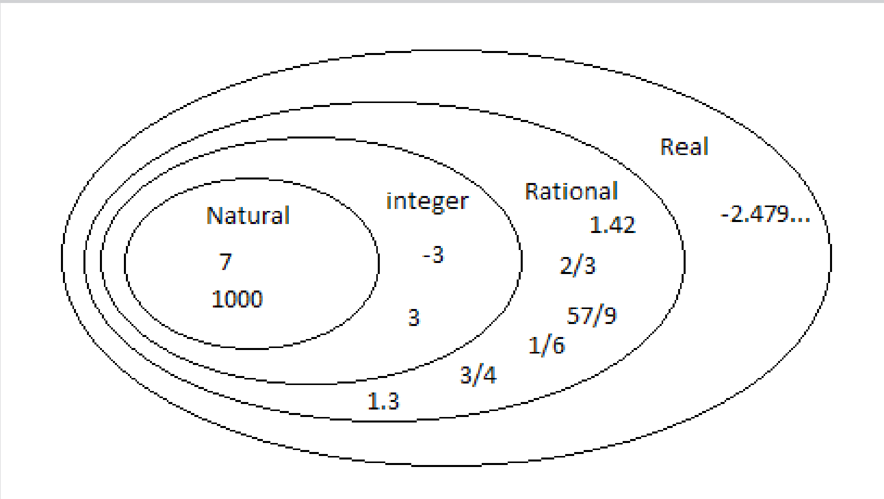
2. a. $\{7,1000\}\subset\mathbb{N}$

   b. $\{-\sqrt{9},7,1000\}\subset\mathbb{Z}$

   c. $\left\{-2.479,\frac{2}{3},\frac{57}{9},\frac{1}{6},\frac{3}{4},1.32,1.42,-\sqrt{9},7,1000\right\}\subset\mathbb{Q}$

   d. $\left\{-2.479,\frac{2}{3},\frac{57}{9},\frac{1}{6},\frac{3}{4},1.32,1.42,-\sqrt{9},\pi,7,1000\right\}\subset\mathbb{R}$

   e. Show the relationship between these sets of numbers is shown below



Lesson 2: Operations and properties on numbers

a) Learning objectives:

- Carry out mathematical operations on numbers
- Explore properties on numbers

b) Teaching resources:

Manila papers, flash card, student's book and other Reference textbooks to facilitate research.

c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good background on concepts of Natural, rational, integers, irrational and real numbers learnt in previous lesson or learnt in ordinary level.

d) Learning activities

- Organize students into small groups;
- Invite them to do activity 1.1.2. found in the student's book;
- Through group discussion invite students to do all questions of the given activity 1.1.2. and visit each group to check if every student is engaged;
- Invite group representatives to present their findings, then help all students to make content summary;
- Use different probing questions and guide students to explore examples and the content given in the student's book to perform operations in the set $\mathbb{N}, \mathbb{Q}, \mathbb{Z}$, $\mathbb{R}$ and guide them to highlight the corresponding properties;
- After this step, guide students to do the application activity 1.1.2., assess their competences and evaluate whether lesson objectives were achieved.

Answers of activity 1.1.2.

1. Operations on number in sets $\mathbb{N}$, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$

Operation/ set	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
Addition	$a+b\in\mathbb{N}$ Example $1+4=5\in\mathbb{N}$	$a+b\in\mathbb{Z}$, Example, $(-5)+8=3\in\mathbb{Z}$	$a + b\in\mathbb{Q}$ Example $\frac{1}{3}+\frac{1}{2}=\frac{5}{6}\in\mathbb{Q}$	$a + b\in\mathbb{R}$ Example $25+3=28\in\mathbb{R}$
Subtraction	$a-b\notin\mathbb{N}$ Example $1-4=3\notin\mathbb{N}$	$a-b \in \mathbb{Z}$ Example $2-1=1\in \mathbb{Z}$	$a - b\in\mathbb{Q}$ Example $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}\in\mathbb{Q}$	$a - b\in\mathbb{R}$ Example $25-3=22\in\mathbb{R}$

Multiplication	$a \times b \in \mathbb{N}$ Example $2 \times 3 = 6 \in \mathbb{N}$	$a \times b \in \mathbb{Z}$ Example $-3 \times 2 = -6 \in \mathbb{Z}$	$a \times b \in \mathbb{Q}$, Example $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$	$a \times b \in \mathbb{R}$, Example $-3 \times 2 = -6 \in \mathbb{R}$
Division	$a \div b \notin \mathbb{N}$, Example $2 \div 3 \notin \mathbb{N}$	$a \div b \notin \mathbb{Z}$, Example (-3)÷(-6)=½, is not an integer.	$a \div b \in \mathbb{Q}$ Example $\frac{2}{3} \div \frac{1}{5} = \frac{10}{3} \in \mathbb{Q}$	$a \div b \in \mathbb{R}$ Example $6 \div 2 \in \mathbb{R}$

2. The given $a, b \, and \, c$, in each set of $\mathbb{N} \, and \, \mathbb{R}$ investigate the following properties and use

Property/set	$\mathbb{N}$	$\mathbb{R}$
Closure property for addition and subtraction	$a + b \in \mathbb{N}$, $\mathbb{N}$ is **closed** under addition $a + b \notin \mathbb{N}$, $\mathbb{N}$ is **not closed** under subtraction	*a + b*, Closure property under addition *a–b*, Closure property under addition
Distributive property for addition	$a \times (b + c) = ab + ac$ multiplication is distributive over addition.	$a \times (b + c) = ab + ac$ multiplication is distributive over addition.
Commutative property	$a + b = b + a$, addition is **commutative** in $\mathbb{N}$. $a \times b = b \times a$ multiplication is **commutative** in $\mathbb{N}$	*a + b = b+ a*, Commutative property under addition in $\mathbb{R}$ $a \times b = b \times a$, multiplication is **commutative** in $\mathbb{R}$

e) Answers of application activity 1.1.2.

1. a. $3(2x - 5) = 6x - 15$, Distributive property of multiplication over subtraction has been expressed.

   b. $(0.08 + 0.12) + \frac{1}{2} = 0.8 + \left(0.12 + \frac{1}{2}\right)$ associativity property under addition has been expressed.

c. $(3\times5)\times2=3\times(5\times2)$ but $(3\div5)\div2\neq3\div(5\div2)$ associativity property for multiplication has been expressed but it cannot be applied on division (since division is not associative).

d. $\pi-2\neq2-\pi$ but $\pi+2=2+\pi$ subtraction is not commutative but for addition, commutativity property has been applied

2. Closure Property under division for real numbers is not satisfied. Note that 0 itself is a rational number (0 = 0/1). So 3 ÷ 0 is a "rational being divided by a rational".

But the result violates the definition of rational form $\frac{p}{q}$, $where\, q\neq0$.

3. Properties and examples

Property/set	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
Closure property for addition and subtraction	$a+b\in\mathbb{N}$, $\mathbb{N}$ is **closed** under addition	Closure property under addition: $a+b\ \in\ \mathbb{Z}$ Example 4+3=7 $\in\ \mathbb{Z}$ Closure property under subtraction, $a-b\in\mathbb{Z}$	$a\ +\ b\in\mathbb{Q}$ $\mathbb{Q}$ is **closed** under addition	*a* + *b*, Closure property under addition *a* − *b*, Closure property under addition
Identity property	The identity element for $\mathbb{N}$ under addition is zero. $\forall n\in\mathbb{N}:0+n=n\in\mathbb{N}$ and 1 is the identity element of multiplication for $\mathbb{N}$	Identity property: $a\times1=a=1\times a$ hence, 1 is called the multiplicative identity for multiplication. The identity element for $\mathbb{Z}$ under addition is zero. $z\in\mathbb{Z}:0+z=z\in$	1 is the multiplicative identity for $\mathbb{Q}$. If a/b is any rational number, then $\frac{a}{b}\times1=1\times\frac{a}{b}=\frac{a}{b}$ Example: $\frac{5}{3}\times1=1\times\frac{5}{3}=\frac{5}{3}$	0 is additive identity Example, $6+0=6\in\mathbb{R}$ 1 is multiplicative identity Example, $6\times1=6\in\mathbb{R}$

Distributive property for addition	$a\times(b+c)=ab+ac$ multiplication is distributive over addition.	Distributive $a\times(b+c)=a\times b+a$ Example: $2(1+4)=(2\times 1)+(2\times 4)$	Distributive property, $a\times(b+c)=(a\times b)+(a\times c)$	$a\times(b+c)=ab+ac$ multiplication is distributive over addition, example $2(3+5)=2(3)+2(5)$ $=6+10=16$
Commutative property	$a+b=b+a$, addition is commutative in $\mathbb{N}$. $a\times b=b\times a$, multiplication is commutative	Commutative property under addition: $a+b\ =b+a$ $a\times b=b\times a$, multiplication is commutative	$a+b=b+a$ Commutative property under addition $a\times b=b\times a$, multiplication is commutative	$a+b=b+a$, Commutative property under addition. Example: 4+7=7+4=11 $a\times b=b\times a$, multiplication is commutative

Lesson 3: Percentages and ratios

a) Learning objective:

Apply percentages and ratios to solve financial, production and economical related problems

b) Teaching resources:

Manila papers, flash card, digital technology including calculators, student's book and other Reference textbooks to facilitate research.

c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good background on concept of rational, fractions and real numbers learnt in previous lesson or learnt in ordinary level.

d) Learning activities

- Ask the students to do activity 1.1.3 in pairs
- Move around in the class to verify students' progress over the work.
- Have a few groups with different activities present their answers to the whole class.

- As a teacher, harmonize students' insights by insisting on how to write a decimal as a percentage, a fraction, and vice versa.
- Ask students to brainstorm the use of percentages in real life experiences: bank, student grades, various exams, research reports, statistics from local administrative readers, etc.
- Use different probing questions and guide students to explore examples and content from the student book to solve related exercises and problems
- After this step, guide the students to complete application activity 1.1.3. conduct, assess their competences and assess whether the teaching objectives have been achieved.

Answers of activity 1.1.3.

1) a) $60:100=\frac{60}{100}=\frac{6}{10}=\frac{3}{5}$

b) $\frac{3}{5}\times 100=60\%$

c) $\frac{60}{100}=\frac{3}{5}=0.6$

2) $\frac{1}{3}=0.3333333333$ and $\frac{22}{7}=3.1428571429$

Yes, we can present the above decimals in the form of percentage as

$\frac{1}{3}\times 100=33.33333\%\approx 33.33\%$

$\frac{22}{7}\times 100=314.286\ \%$

3) First, find out how many students did not pass

Let x be percentage of students who did not pass test

Students who did not pass: $24-18=6$

Then, $x\%$ of 24 is equal to 6 students

Let find $x\%$, $x\%\ of\ 24=6$

$x=\frac{6}{24}=0.25=\frac{25}{100}=25\%$

Then, 25% of students did not pass the test.

4) Yes, the basic understanding of percentages and ratios concepts can help bank managers, local leaders and shop keepers improve decision-making process and professional success. With a better understanding of how their organization measures financial performance, they can take steps to add value to their day-to-day activities.

Note: Students will give different answers and teacher harmonize them accordingly.

e) Answers to application activity 1.1.3

1) The number correct answers is 80% of 20 or $\frac{80}{100}\times 20$

$\frac{80}{100}\times 20 = 0.80\times 20 = 16$

Number of missed questions is $20-16=4$

2) Question 1 and 2 are corrected in the same way: New value(N)=1650

Original value (O)=1500

Percent increase $x=\frac{N-O}{O}=\frac{1650-1500}{1500}=0.1=10\%$

Therefore, there is a 10% increase

Lesson 4: Venn diagrams and operations on sets of numbers

a) Learning objective:

Use Venn diagrams and operations on sets of numbers in modeling and solving production, financial and economical related problems.

b) Teaching resources:

Manila papers, flash card, digital technology including calculators, student's book and other reference textbooks to facilitate research.

c) Prerequisites/Revision/Introduction:

- Students will do well in this lesson if they have enough skills on the Venn diagram from the previous lesson
- Students will perform better in this lesson if they have knowledge of set operations learned in Senior One.

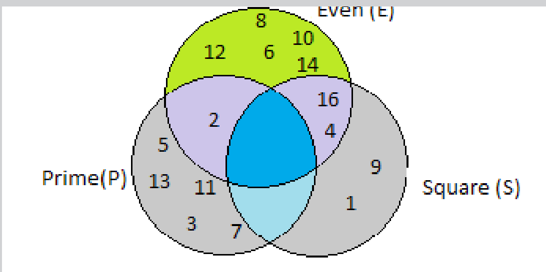
d) Learning activities

- Ask the students to do activity 1.1.4 in pairs. on percent and related problems
- Move around in the class to verify students' progress over the work.
- Have a few groups with different activities present their answers to the whole class.
- As a teacher, harmonize students' insights by insisting on how the given information is represented in the Venn diagram
- Use a variety of probing questions to guide students to examine examples and content of using operations on sets: intersection, union, difference, symmetric difference and complement of set, and guide them to highlight the appropriate properties.
- After this step, direct the students to work on application activity 1.1.4. conduct, assess their competences and assess whether the teaching objectives have been met.

Answers of activity 1.1.4.

1. Venn diagram is a diagram used to represent all possible relations of different sets. Typically, it represented by intersecting and non-intersecting circles but other closed figures like squares, rectangles, etc…. may be used to represent it.

2.



3. a) All students who were absent in the Auditing class: A'

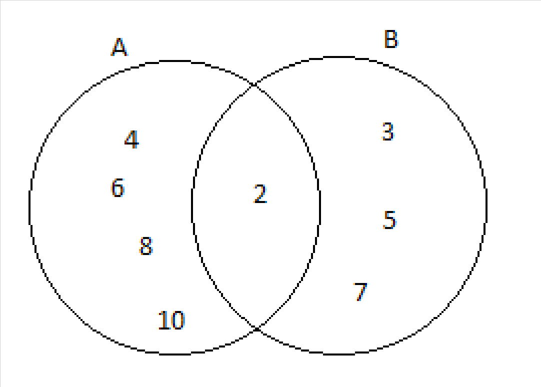
   b) All students who were present in at least one of the two classes: $A \cup B$

   c) All the students who were present for both Auditing as well as Taxation classes: $A \cap B$

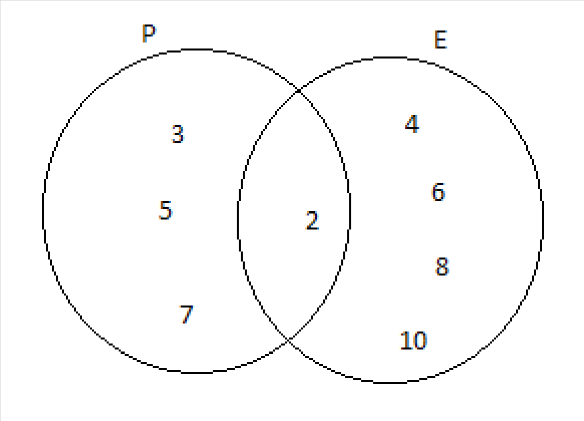
   d) All the students who have attended only the Auditing class and not the Taxation class: $A - B$

e) Answers for application activity 1.1.4.

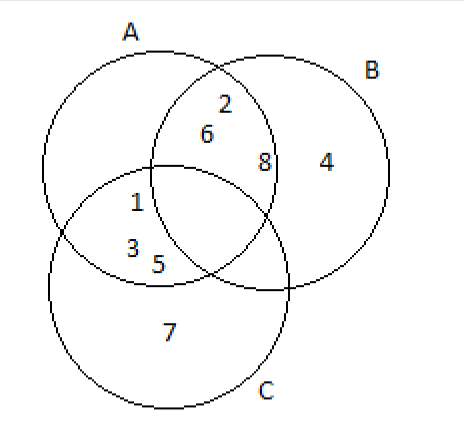
1. Representation of given set A and B on Venn diagram



2. The Venn diagram representing the situation is:



3. $A=\{1,2,3,5,6,8\}$; $B=\{2,4,6,8\}$; $C=\{1,3,5,7\}$



a) $B\cap C=\{\phi\}$;

b) $A\cap C=\{1,3,5\}$

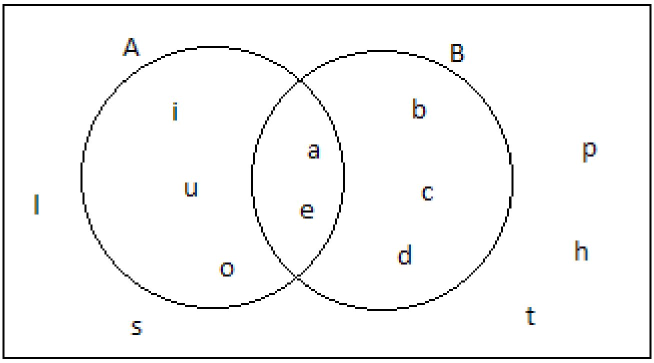
c) $A\cap B=\{2,6,8\}$

d) $B\cup C=\{1,2,3,4,5,6,7,8\}$

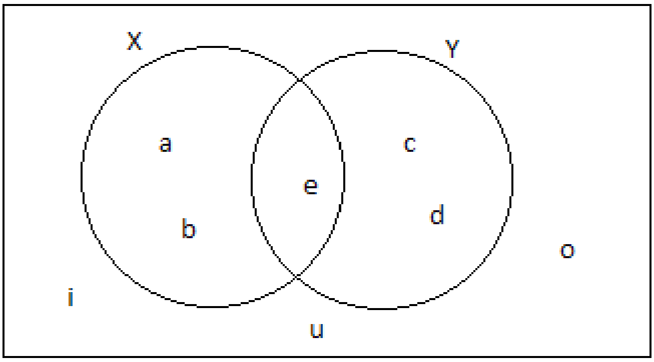
e) $A\cup B=\{1,2,3,4,5,6,8\}$

4. a) $A=\{i,u,o,a,e\}$; b) $B=\{a,b,c,d,e\}$; c) $A-B=\{i,u,o\}$;

d) $B-A=\{b,c,d\}$; e) $A\Delta B=\{i,u,o,b,c,d\}$;



5. $U=\{a,\ e,\ i,\ o,\ u,\ c,\ d\}$; $X=\{a,\ b,\ e\}$ *and* $Y=\{c,\ d,\ e\}$



a) $X\cap Y=\{e\}$; b) $(X\cap Y)'=\{a,b,c,d\}$;

c) $X\cup Y=\{a,b,c,d,e\}$; d) $(X\cup Y)'=\{i,u,o\}$

At the end of this lesson, you can give other many possible exercises as remedial and consolidation activities of this lesson.

Lesson 5: Applications of set theory in finance, production and economics related problems

a) Learning objective:

Apply set operations in solving production, financial and economical related problems

b) Teaching resources:

Compass, Manila paper, Markers, Calculators, student's book and other Reference textbooks to facilitate research, wall charts and wall maps, Mathematical models and Internet connection where applicable.

c) Prerequisites/Revision/Introduction:

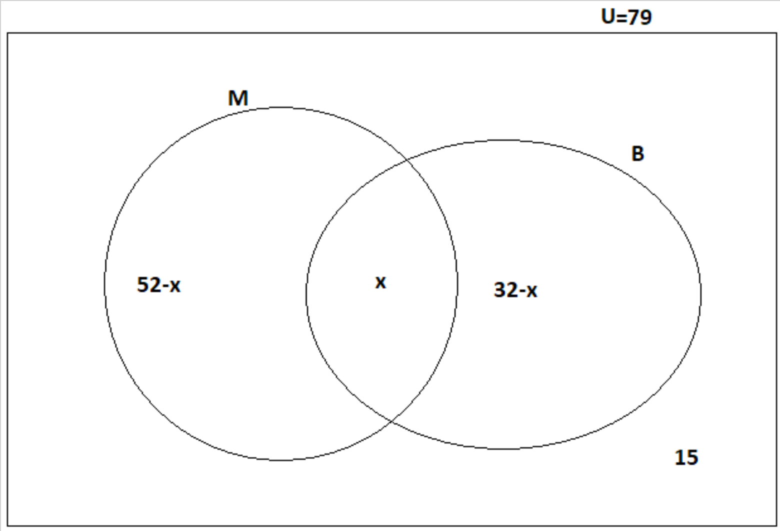
- Students will do well in this lesson if they have enough skills on the Venn diagram, set of numbers and set operations from the previous lessons
- Students will learn better this lesson if they have a good background on basic concepts of algebra and sets of numbers (definition, notation, operations and properties)

d) Learning activities

- Organize the students into small groups;
- Provide clear instructions and introduce the activity by guiding students.
- Ask students to use the student's book to discuss all questions of the activity 1.1.5.
- Switch to each group to ensure all students are actively participating
- Call upon groups to present their findings and harmonize their answers;
- Use various probing questions to guide students to explore examples and content of how some practical word problems can be illustrated and solved using Venn diagrams.
- After this step, guide the students to complete the application activity 1.1.5., assessing their competences and evaluating whether the lesson objectives have been met.

Answers of activity 1.1.5.

1. a) and b) let M represent those who bought milk and B represent those who bought bread: $n(\mathrm{M}) = 52; \mathrm{n}(\mathrm{B} = 32)$; $\mathrm{n}(\mathrm{U}) = 79$ let those who bought both milk and bread be represented by x that is $x = n(\mathrm{M} \cap \mathrm{B})$;



$$52 - x + 32 - x + x + 15 = 79 \Rightarrow x = 20$$

i) Customers who bought milk and bread are 20

ii) Customers who bought bread only are $32 - 20 = 12$

iii) Customers who bought milk only are $52 - 20 = 32$

c) The simplest method used in (a) and (b) above is Venn diagram to represent the situation because it clarifies the situation in a simple way.

2. As 55 students like Mathematics; $x + x + 5 + 20 = 55 \Rightarrow 2x = 30 \Rightarrow x = 15$

   a) 15 students like all subjects

   b) The total number of students is $= U = 10 + 60 + 20 + 15 + 15 + 5 = 125$

   c) Students who like Auditing and taxation only are =0 (no one)

e) Application activity 1.1.5.

1. let A be set of persons who got medals in rental income taxes

   B be the set of persons who got medals in taxes on property

   C be the set of persons who got medals in taxes on goods and

   Given, $n(A) = 36, n(B) = 12\ n(C) = 18,$
   $n(A \cup B \cup C) = 45, \quad n(A \cap B \cap C) = 4$

   We know that number of elements belonging to exactly two of the three sets A, B, C

$$= n(A\cap B) + n(B\cap C) + n(A\cap C) - 3n(A\cap B\cap C)$$

$$= n(A\cap B) + n(B\cap C) + n(A\cap C) - 3\times 4 \quad \text{(i)}$$

$$n(A\cup B\cup C) = n(A) + n(B) + n(C) - n(A\cap B) - n(B\cap C) - n(A\cap C) + n(A\cap B\cap C)$$

Therefore,

$$n(A\cap B) + n(B\cap C) + n(A\cap C) = n(A) + n(B) + n(C) + n(A\cap B\cap C) - n(A\cup B\cup C)$$

From (i) the required number is given by:

$$= n(A) + n(B) + n(C) + n(A\cap B\cap C) - n(A\cup B\cup C) - 12$$

$$= 36 + 12 + 18 + 4 - 45 - 12 \quad \Rightarrow 70 - 57$$

$$= 13$$

The received medals in exactly two of these categories are 13

2. Let x be the number of customers. The customers who bought the three customers are 15

i) The total number of customers is given by $60 + 10 + 15 + 20 + 20 = 125$

ii) Customers who bought Cell phones (C) and fruits (F) only = 0.

By the end of this lesson the teacher should be able to give other many possible exercises as remedial and consolidation activity of this lesson.

Lesson 6. Indices/powers/exponents

a) Learning objective:

Solve mathematical and financial related problems using properties of indices

b) Teaching resources:

Manila paper, Markers, Calculators, student's book and other Reference textbooks to facilitate research, Mathematical models and Internet connection where applicable

c) Prerequisites/Revision/Introduction:

- Students will do well in this lesson if they have enough skills on the sets of numbers, basic concepts of arithmetic
- Students will do well in this lesson by briefly reviewing powers of a real number learned in S2 and S3.

d) Learning activities

- Organize the students into small groups;
- Provide clear instructions and introduce the activity by guiding students.
- Ask students to use the student's book to discuss all questions of the activity 1.2.1.
- Switch to each group to ensure all students are actively participating
- Call upon groups to present their findings and harmonize their answers;
- Use various probing questions to guide students to explore examples and content related to indices/exponent/powers
- Ask students to give examples of where applying properties of powers to solve real world problems focusing in finance or economics
- After this step, guide the students to do the application activity 1.2.1., assessing their competences and evaluating whether the lesson objectives have been met

Answers of activity 1.2.1.

1. The given paper has the form of square, $Area = x.x = x^2$

   $Area = 20cm \times 20cm = (20cm)^2 = 400cm^2$

2. $Volume = a.a.a = a^3$

   $volume = 3dm.3dm.3dm = (3dm)^3 = 3^3 dm^3 = 27dm^3$

e) Answers of application activity 1.2.1.

1) simplification

   a) $x^3 x^2 = x^{3+2} = x^5$

   b) $(xy^3)^2 + 4x^2y^6 = x^2y^6 + 4x^2y^6 = 5x^2y^6$

   c) $\frac{6xy^2}{3xy} = \frac{2y}{1} = 2y$

   d) $\frac{ab}{a^3} - \frac{a^3b^2}{a^5b} = \frac{a^3b^2 - a^3b^2}{a^5b} = 0$,

   e) $\frac{yx}{4xy} = \frac{1}{4}$

2) Exponents/powers are useful in Economics, Accounting, and Finance especially in finding compound interest. Students will provide different answers. Verify them using other reference books.

Lesson 7. Surds

a) Learning objective:

Appreciate the importance of surds in solving financial related problems

b) Teaching resources:

Digital materials including calculator, Manila paper, Markers, Calculators, student's book and other Reference textbooks to facilitate research, Mathematical models and Internet connection where applicable

c) Prerequisites/Revision/Introduction:

- Students will do well in this lesson if they have enough skills on the sets of numbers, and basic concepts of arithmetic
- Students will do well in this lesson by briefly reviewing powers and surds of a real number learned in ordinary level.

d) Learning activities

- In group discussions, invite students to do activity 1.2.2A. in student's book on surds and related problems.
- Use gallery walk, students share their answers to others by rotating and ask support on challenging points they faced in their group.
- Move around to see student's progress in their respective groups.
- Invite groups with different working steps to present their answers then, harmonize the presented answers.
- After doing this step, use different questions and guide students to discover properties of powers and examples.
- Guide students to deal with operations on surds through examples and remember to highlight different rules: simplification of surds and rationalizing a denominator.
- After this, invite students to do application activity 1.2.2A., assessing their competences and evaluating whether the lesson objectives have been met

Answers for Activity 1.2.2.1

1) a) $(81)^{\frac{1}{2}} = 9$ b) $(216)^{\frac{1}{3}} = 6$ c) $(-27)^{\frac{1}{3}} = -3$ d) $(16)^{\frac{1}{4}} = 2$

2) a) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\frac{\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)}{\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)}=5-2\sqrt{6}$

b) $\frac{4-\sqrt{6}}{\sqrt{2}}=\frac{\left(4-\sqrt{6}\right)\left(\sqrt{2}\right)}{\left(\sqrt{2}\right)\left(\sqrt{2}\right)}=2\sqrt{2}-\sqrt{3}$

e) Answers of application activity 1.2.2.1

a. $\frac{2\sqrt{2}}{4+3\sqrt{3}}=\frac{\left(2\sqrt{2}\right)\left(4-3\sqrt{3}\right)}{\left(4+3\sqrt{3}\right)\left(4+3\sqrt{3}\right)}=\frac{2\sqrt{2}\left(4-3\sqrt{3}\right)}{-11}=\frac{-2\sqrt{2}\left(4-3\sqrt{3}\right)}{11}$

b. $\frac{a-\sqrt{b}}{\sqrt{d}}=\frac{(a-\sqrt{b})}{\sqrt{d}}\times\frac{\sqrt{d}}{\sqrt{d}}=\frac{a\sqrt{d}-\sqrt{b}\sqrt{d}}{\sqrt{d}\sqrt{d}}=\frac{\sqrt{d}\left(a-\sqrt{b}\right)}{d}$

c. $\frac{3\sqrt{3}+2\sqrt{2}}{1+2\sqrt{2}}=\frac{3\sqrt{3}+2\sqrt{2}}{1+2\sqrt{2}}\times\frac{1-2\sqrt{2}}{1-2\sqrt{2}}=\frac{3\sqrt{3}-6\sqrt{6}+2\sqrt{2}-8}{1-8}$

Lesson 8. Absolute value

a) Learning objective:

Appreciate the importance of absolute value in solving financial related problems

b) Teaching resources:

Digital materials including calculator, Manila paper, Markers, Calculators, student's book and other Reference textbooks to facilitate research, Mathematical models and Internet connection where applicable

c) Prerequisites/Revision/Introduction:

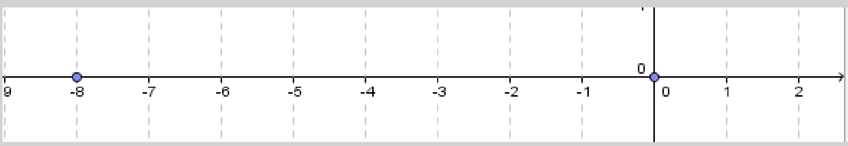
- Students will do well in this lesson if they have enough skills on the sets of numbers, and basic concepts of arithmetic
- Student-teachers will perform well in this unit if they make a revision on the absolute value of a number learnt in S2 and S3.

d) Learning activities

- In group discussions, invite students to do activity 1.2.2B. in student's book on Properties and operations on absolute value
- Use gallery walk, students share their answers to others by rotating and ask support on challenging points they faced in their group.
- Move around to see student's progress in their respective groups.
- Invite groups with different working steps to present their answers then, harmonize the presented answers.
- After doing this step, use different questions and guide students to discover Properties on absolute value
- Together with students, invite them to read from content summary in student book. With clear examples, insist on the meaning of $|x| = \begin{cases} x, \text{if } x \geq 0 \\ -x, \text{if } x < 0 \end{cases}$.
- Together with students, do the same procedure above on properties of absolute value.
- Guide students to deal with Properties and operations on absolute value through examples
- After this, invite students to do application activity 1.2.2B., assessing their competences and evaluating whether the lesson objectives have been met

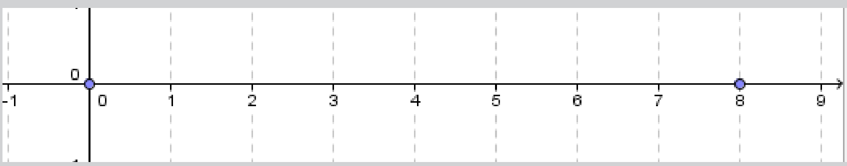
Answers for activity 1.2.2.2.

1. a.



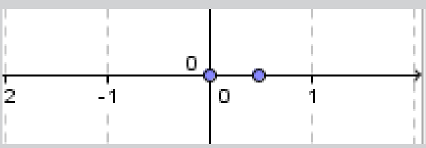
There are 8 units between 0 and -8.

b.

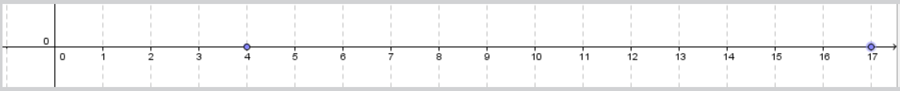


There are 8 units between 0 and 8

c.



There are 0.5 units between 0 and $\frac{1}{2}$.

d. 

There are 13 units between 4 and 17.

2. No, Negatives are usually associated with displacement, where one direction is considered positive and the opposite is considered negative. Distance is just how far you have travelled. Starting from 0 m and increasing until you stop.

Students may give different solution, as teacher harmonize them.

e) Application activity 1.2.2.2.

1) a) $|x| = 6 \Rightarrow x = 6 \; or \; x = -6 \;\; \Rightarrow S = \{-6; 6\}$

b) $|x-3| - 4 = 2$

$$|x-3| - 4 = 2$$
$$x - 3 = 2 + 4$$
$$x = 9$$

or

$$-(x-3) - 4 = 2$$
$$-x + 3 - 4 = 2$$
$$-x = 3$$
$$x = -3$$

Therefore, $S = \{-3; 9\}$

2) To simplify

a) $|-4||-5|$ =20 b) $|-7| + |4|$ = 11 c) $-|4 \times 6|$ = -24 d) $-|-6+8|$ = -2

3) Distance between OA is given by $d(\mathrm{O}, \mathrm{A}) = |A - 0|$.

Distance between O and A is $|-4-0| = |-4| = 4$

Distance between C and B is $d(\mathrm{B}, \mathrm{C}) = |C - B|$

$$d(C, B) = \left|-\frac{5}{2} - 2\right| = \frac{9}{2} = 4.5$$

Lesson 9. Decimal and Naperian logarithms

a) Learning objective:

Transform a logarithmic expression to equivalent power or radical form and vice versa

b) Teaching resources:

Digital materials including calculator, Manila paper, Markers, Calculators, student's book and other Reference textbooks to facilitate research, Mathematical models and Internet connection where applicable

c) Prerequisites/Revision/Introduction:

- Students will perform well in this unit if they are enough skilled in properties of powers/indices/exponents acquired in mathematics for ordinary level or in previous covered unit.

d) Learning activities

- Invite students to work in groups and do the activity 1.2.3. from the student' book;
- As students are working on the given activity, circulate to each group for guidance
- Ask the groups to share their answers with other groups and allow them to share the challenging points they faced in their groups.
- As Teacher, try to harmonize students' findings;
- Ask them different questions that will lead them to discover the meaning of the decimal logarithm of a number written to the power of 10.
- After attempting different examples, help them to formulate the decimal logarithm of a number and establish how to find it. Highlight the properties of decimal logarithms, supported with examples, and show that $\log x$ is the same as $\log_{10} x$, where 10 is the base and is only defined for all positive real numbers. Insist that in general notation we do not write this base for the decimal logarithm. In the notation $y = \log x$, x is supposed to be the antilogarithm of y.
- Ask students probing questions to discover the properties of the natural logarithm, and give examples on each property
- Guide students to find more real-life examples that can be solved with the intervention of decimal and natural logarithms.

- After this, invite students to do application activity 1.2.3., assessing their competences and evaluating whether the lesson objectives have been met

Answers for activity 1.2.3.

1. 1. The number requested is the exponent of 10 in the following expressions

   1. $1=10^0$ 2. $10=10^1$ 3. $100=10^2$ 4. $1000=10^3$ 5. $10000=10^4$
   6. $100000=10^5$

2. To find the number x in $x^3=64$, we can equalize it with $x^3=64=4^3$ and deduce that $x=4$

e) Answers for application activity 1.2.3.

1. (a). $a>b$ (b) $a=b$ (c) $a<b$
2. (a) 2.17 (b) 0.66 (c) 0.30
3. a) $co\log 100=-\log 100=-2$
   b) $co\log 42=-\log 42=-1.623$
   c) $co\log 15=-\log 15=-1.176$
4. $\ln\frac{1}{\sqrt{e}}=\ln e^{-1/2}=-\frac{1}{2}\ln e=-\frac{1}{2}$
5. $\frac{1}{3}\ln(x-1)-\frac{1}{2}\ln(x+1)=\ln\sqrt[3]{x-1}-\ln\sqrt{x+1}=\ln\frac{\sqrt[3]{x-1}}{\sqrt{x+1}}$

Lesson 10. Applications of exponents and logarithms

a) Learning objective:

Use powers and logarithms to optimize simple problems related to production, finance and economics.

b) Teaching resources:

Digital materials including calculator, Manila paper, Markers, Calculators, student's book and other Reference textbooks to facilitate research, Mathematical models and Internet connection where applicable.

c) Prerequisites/Revision/Introduction:

Students will perform well in this unit if they have enough competences on the basic concepts of algebra acquired in ordinary level and the content of previous lessons in this unit.

d) Learning activities

- Invite students to work in groups and do the activity 1.2.4. from the student' book about research in library or internet on the use of logarithms in Economics and Finance
- Ask each group to make summary and prepare a presentation of findings with other groups.
- Invite group members to present their answers to the whole class;
- As a teacher, harmonize the results of the presentation. When harmonizing, insist on showing the students that math is needed everywhere, especially when it comes to economics and finance.
- Ask students different probing questions that will lead them to discover the use of exponents and logarithms especially in economics, production and finance.
- After attempting different examples, help them to solve given examples in student' s book
- Guide students to find more real-life examples that can be solved with the intervention of exponents and logarithms especially in economics, production and finance
- After this, invite students to do application activity 1.2.4., assessing their competences and evaluating whether the lesson objectives have been met.

Answers for activity 1.2.4.

1. Exponents and logarithms are very important for economist and accountant. For example, they can use them in calculation of compound interest and Final value of an investment.

   The compound interest: Considering the case in which P is invested in a bank for n interest periods per year at the rate r expressed as a decimal. The accumulated amount A after the time t (number of years P is invested) is given by:

   $A=P\left(1+\frac{r}{n}\right)^{nt}$ Where n is the number of interest periods per year, r is the interest rate expressed as decimal, A is the amount after t years. From the formula, economist and accountant can apply logarithms to find out t years of an investment. $t=\frac{\ln A}{n\ln P\left(1+\frac{r}{n}\right)}$

Calculating the final value of an investment: Consider an investment at compound interest where: A is the initial sum invested, F is the final value of the investment, i is the interest rate per time period (as a decimal fraction) and n is the number of time periods. The formula is given by $F = A(1+i)^n$

then $n = \dfrac{\ln(1+i)}{\ln F}$

Note:

When the interest rate is compounded per year, $A = P(1+r)^n$ where r is expressed as a decimal for example $r = 9\% = 0.09$.

2. For the given problem, we'll use the compound Interest formula,

   $F = P(1+i)^t$ where F is the final value, P the initial value of investment.

   $$100000 = 70000\left(1+\frac{11}{100}\right)^t$$

   $$10 = 7(1.11)^t$$

   $$t = \frac{\log\left(\frac{10}{7}\right)}{\log(1.11)} = 3.41 \quad \text{Or } t = \frac{\ln\left(\frac{10}{7}\right)}{\ln(1.11)} = 3.41$$

Therefore, the time needed is 3.41 years

3. From the formula $F = P(1+i)^t$, Mr. Mateso can discover that the money increase due to compounded interest given by $i = \left(\frac{F}{P}\right)^{\frac{1}{t}} - 1$,

   $$i = \left(\frac{8{,}000{,}000}{200}\right)^{\frac{1}{3}} - 1 \Rightarrow i = (40{,}000)^{\frac{1}{3}} - 1$$

e) Answers for Application activity 1.2.4.

1) An initial investment of £50,000
   Ending amount £56,711.25 and time (t) = 2 years.

   $$\frac{F}{A} = \frac{56711.25}{50000} = 1.134225$$

   $$i = \sqrt[n]{\left(\frac{F}{A}\right)} - 1 = \sqrt[2]{1.13455} - 1 = 1.065 - 1 = 0.065$$

   $i = 6.5\%$

   The interest rate has been applied is 6.5%

2. Let us use the following formula,

$$A = P\left(1+\frac{r}{n}\right)^{nt} \quad then, 2000 = 1000\left(1+\frac{0.0635}{12}\right)^{12t}$$

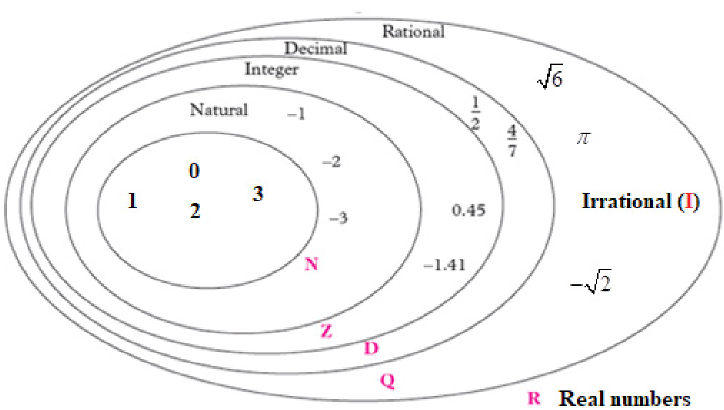
$$\frac{\log 2}{12\log\left(1+\frac{0.0635}{12}\right)} = t \quad \Rightarrow t = 10.9\,years$$

Therefore, the account will double in 10.9 years

1.6. Summary of the unit 1

A set is a collection of objects (called members or elements) that is regarded as being a single object. To indicate that an object *x* is a member of a set *A* one writes $x \in A$, while $x \notin A$ indicates that *x* is not a member of *A*. For example, the set given by "prime numbers less than 10" can be given by {2, 3, 5, 7}. In principle, any finite set can be defined by an explicit list of its members, but specifying infinite sets requires a rule or pattern to indicate membership; for example, {0, 1, 2, 3, 4, 5, 6, 7, …} indicates that the list of natural numbers N goes on forever. The empty (null) set, symbolized by {} or Ø, contains no elements at all. Nonetheless, it has the status of being a set. We have different set of numbers in which has relationship as follows: $\mathbb{N} \subset \mathbb{Z} \subset D \subset \mathbb{Q} \subset \mathbb{R}$.

They can be summarized in the following figure with examples of numbers in each set:



In accounting option, we need to calculate percentages and ratios as the percentage can increase or decrease then, we can calculate percentage increase by dividing the increase by original number and finally we multiply the answer by 100. While, to calculate percentage decrease, we work out difference first, then we divide the obtained decrease by original number and multiply answer by 100. In addition, the properties of logarithms help us to solve economics, finance, and production related problems. The following properties applied:

a. Product rule: $ln(x.y) = ln(x)+ln(y)$.

b. Quotient rule: $ln(\frac{x}{y}) = ln(x)-ln(y)$.

c. Power rule: $lnx^{y} = ylnx$.

d. $ln(x) is\ undefined\ when\ x \leq 0$ and $ln(0) is\ undefined$

e. $ln1 = 0$

f. $\ln e = 1$

Note: The rules for using natural logarithms are the same as for decimal logarithms to any other base.

From the formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ of compound interest, both natural and decimal logarithms can be applied to find out the time t years of any investment. Natural and decimal logarithms also can be used to calculate growth rate. For more information, refer to unit 5.

1.7. Additional Information for Teacher

About set of numbers emphasize on set of natural numbers as follows:

– Ones defines the set $\mathbb{N}$ of the natural numbers as the set $\mathbb{N}$:= {0, 1, . . . , n, . . .} where 0 included in this set of numbers. The axiom of infinity will be needed to guarantee the existence of the set $\mathbb{N}$ of all natural numbers. Within the axiomatic of Zermelo and Fraenkel the natural numbers are defined as special sets.

$$0 = n(\varnothing)$$
$$1 = n\{0\} = 0 + n(\varnothing)$$
$$2 = n(\{1\} = 1 + n\{1\}$$
$$3 = n(\{0,1,2\}) = 2 + n(\{2\})$$
$$\ldots$$
$$n+1 = n(\{0,1,2,3,\ldots n\}) = n + (\{n\})$$

Therefore, set $\mathbb{N}$:= {0, 1, . . . , n, . . .}

Many properties of the natural numbers can be derived from the five Peano axioms

1. 0 is a natural number.

2. Every natural number has a successor which is also a natural number.

3. 0 is not the successor of any natural number.

4. If the successor of {\displaystyle x} x equals the successor of {\displaystyle y} y, then {\displaystyle x} x equals {\displaystyle y} y.

5. The axiom of induction : If a statement is true of 0, and if the truth of that statement for a number implies its truth for the successor of that number, then the statement is true for every natural number.

– Others argue that Natural numbers are the numbers that are used for counting and are a part of real numbers. The set of natural numbers include only the positive integers, i.e., $\{1,2,3,4,...\}$ and they argue that the set given by $\{0,1,2,3,4,...\}$ is a set of whole numbers

They denote as follows: $\mathbb{N}$ Set of natural numbers with zero, $\mathbb{N}=\{0,1,2,3,4,....\},0\in\mathbb{N}$ while other denote $\mathbb{N}_0$ natural numbers without zero given by $\mathbb{N}_0=\{1,2,3,4,...\}$

1.8. End unit assessment

Answers

1. $\sqrt[n]{a^{2n+1}}=\left(a^{2n+1}\right)^{1/n}=a^2.a^{1/n}=a^2\sqrt[n]{a}$

2. $\sqrt[4]{(4)^{-3}}.(2)^7+(6)^{-0.2}.(36)^2=(2)^{-3/2}.(2)^7+(6)^{-0.2}.(6)^4$

$\Rightarrow(2)^{-3/2}.(2)^7+(6)^{-0.2}.(6)^4=(2)^{11/2}+(6)^{19/5}=\sqrt{2^{11}}+\sqrt[5]{6^{19}}$

3. a. $2\log_b x+\frac{1}{2}\log_b(x+4)=\log_b x^2+\log_b(x+4)^{1/2}=\log_b\left[x^2(x+4)^{1/2}\right]$

b. $4\log_b(x+2)-3\log_b(x-5)=\log_b\frac{(x+2)^4}{(x-5)^3}$

c. $\log_b\left(\frac{x\sqrt{y}}{z^5}\right)=\log_b x+\frac{1}{2}\log_b y-5\log_b z$

(To obtain answer use quotient, product and power properties respectively)

4. Let A be the set of taxpayers can pay their taxes through cell telephone. B be the set of people of taxpayers can pay their taxes through bank A - B be the set of taxpayers can pay their taxes through cell telephone and not bank. B - A be the set of tax payers can pay their taxes through cell bank and not cell phone.

$$n(A)=n(A-B)+n(A\cap B)\Rightarrow n(A-B)=n(A)-n(A\cap B)$$
$$=72-15=57$$

$$and\ n(B-A)=n(B)-n(A\cap B)=43-15=28$$

Therefore, Number taxpayers can pay their taxes through cell telephone only = 57. Number taxpayers can pay their taxes through bank only = 28

5. Let use the formula of compound interest

$$A=P\left(1+\frac{r}{n}\right)^{nt}\Rightarrow 110,000=30000\left(1+\frac{0.1}{2}\right)^{2t}$$

$$\log\left(\frac{11}{3}\right)=2t\log(1+0.05)\Rightarrow t=\frac{\log\left(\frac{11}{3}\right)}{2\log(1.05)}\approx 13.3$$

1.9. Additional activities

1.9.1. Remedial activities

1. **Simplify the following**

a. $\sqrt{46656}=\sqrt{6^6}=6^3=216$

b. $\sqrt[3]{ab}\times\sqrt[3]{a^2b^2}=\sqrt[3]{a^3b^3}=\sqrt[3]{(ab)^3}=ab$

$$\sqrt{\frac{36}{81}}=\frac{\sqrt{36}}{\sqrt{81}}=\frac{6}{9}=\frac{2}{3}$$

c. $\left(\frac{\sqrt{2}\sqrt{3}}{\sqrt{12}}\right)^3=$

$$\left(\frac{\sqrt{2}\sqrt{3}}{\sqrt{12}}\right)^3=\left(\frac{\sqrt{2\cdot 3}}{\sqrt{12}}\right)^3=$$

$$=\left(\frac{\sqrt{6}}{\sqrt{12}}\right)^3$$

Because we have a root in the numerator and denominator (same degree) we can write them as one single root:

$$\left(\frac{\sqrt{6}}{\sqrt{12}}\right)^3 = \left(\sqrt{\frac{6}{12}}\right)^3$$

Note: the step above is due to the properties of powers because

$$\frac{\sqrt{6}}{\sqrt{12}} = \frac{6^{1/2}}{12^{1/2}} = \left(\frac{6}{12}\right)^{1/2} = \sqrt{\frac{6}{12}}$$

Now we can simplify the radicand's fraction:

$$\left(\sqrt{\frac{6}{12}}\right)^3 = \left(\sqrt{\frac{1}{2}}\right)^3$$

Due to the fact we have a 1 in the numerator and its square root is 1, we are going to write the two roots again:

$$\left(\sqrt{\frac{1}{2}}\right)^3 = \left(\frac{\sqrt{1}}{\sqrt{2}}\right)^3 = \left(\frac{1}{\sqrt{2}}\right)^3$$

We calculate the cube of the quotient:

$$\left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1^3}{\sqrt{2}^3} =$$

$$= \frac{1}{\sqrt{2}\sqrt{2}\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

so by rationalizing we get,

$$\frac{1}{2\sqrt{2}} = \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{4}$$

d. $\left(\sqrt[12]{49}\right)^3 =$

$$\left(\sqrt[12]{49}\right)^3 = 49^{\frac{3}{12}} =$$

$$= 49^{1/4}$$

We have simplified the exponent's fraction. Now we will write the radicand (49) as a power: $49 = 7^2$.

$$49^{\frac{1}{4}} = (7^2)^{\frac{1}{4}} =$$

$$= 7^{\frac{2}{4}} = 7^{\frac{1}{2}} = \sqrt{7}$$

1.9.2. Consolidation activities

1. Rationalise $\frac{\sqrt{3}+\sqrt{7}}{4\sqrt{2}}$

Solution: $\frac{\sqrt{3}+\sqrt{7}}{4\sqrt{2}}=\frac{\left(\sqrt{3}+\sqrt{7}\right)\sqrt{2}}{\left(4\sqrt{2}\right)\sqrt{2}}=\frac{\sqrt{6}+\sqrt{14}}{8}$

2. How does the power rule for logarithms help when solving logarithms with the form $log_b(\sqrt[n]{x})$

Solution:

Any root expression can be rewritten as an expression with rational exponent so that the power rule can be applied, making the logarithm easier to calculate.

Thus $\log_b\left(x^{\frac{1}{n}}\right)=\frac{1}{n}\log_b\left(x\right)$

1.9.3. Extended activities

1) Rationalize

$\frac{\sqrt{2}}{\sqrt{5}-\sqrt{3}}=$

$$\frac{\sqrt{2}}{\sqrt{5}-\sqrt{3}}=\frac{\sqrt{2}\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)}=\frac{\sqrt{10}+\sqrt{6}}{5-3}=\frac{\sqrt{10}+\sqrt{6}}{2}$$

2) Each student in a class of 40 plays at least one game football, volleyball and handball. 18 play football, 20 play handball and 27 play volleyball. 7 play football and handball, 12 play handball and volleyball and 4 play football, volleyball and handball. Find the number of students who play

iii) football and volleyball.

iv) football, volleyball but not handball.

Answers

Let A be the set of students who play football

B be the set of students who play handball

C be the set of students who play volleyball

Therefore, We are given n$\left(A\cup B\cup C\right)$ =40,

n(A)=18, n(B)=20 n(C)=27,

$n(A \cap B) = 7$, $n(C \cap B) = 12$ $n(A \cap C \cap B) = 4$

$$n(A \cup B \cup C)=n(A)+n(B)+n(C)-n(A \cap B)-n(B \cap C)-n(C \cap A)+n(A \cap B \cap C)$$

Therefore, $40=18+20+27-7-12-n(C \cap A)+4$

$40= 69 - 19 - n(C \cap A)$

$40 = 50 - n(C \cap A)$ $n(C \cap A) = 50 - 40$

$n(C \cap A) = 10$

Therefore, Number of students who play football and volleyball are 10.

Also, number of students who play football, volleyball and not handball.

$= n(C \cap A)-n(A \cap B \cap C)$

$= 10 - 4$

$= 6$

3) There are 35 students in auditing class and 57 students in financial management class. Find the number of students who are either in auditing class or in financial management class.
   - When two classes meet at different hours and 12 students are enrolled in both activities.
   - When two classes meet at the same hour.

Solution:

$n(A) = 35$, $n(B) = 57$, $n(A \cap B) = 12$

Let A be the set of students in auditing class.

B be the set of students in financial management class.

i) When 2 classes meet at different hours $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$= 35 + 57 - 12 = 80$

ii) When two classes meet at the same hour,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = n(A) + n(B) = 35+57=92$$

UNIT 2 NUMERICAL FUNCTIONS, EQUATIONS, AND INEQUALITIES

2.1. Key unit competence

Solve production, financial and economical related problems using numerical functions, equations and inequalities

2.2. Prerequisite

Students will perform better in this unit if they have background on:

- Definition of polynomial and linear equation from ordinary level
- Applying operation properties to carry out given operation of polynomials
- Basic concepts of algebra and giving common factor of algebraic expressions;
- Set of numbers (properties and operations)
- The use correctly simple language structure, vocabulary and suitable symbols of mathematics learnt in Ordinary Level;

2.3. Cross-cutting issues to be addressed

- **Inclusive education**: promote the participation of all students while teaching process
- **Peace and value Education**: During group activities, the teacher will encourage students to help each other and to respect opinions of colleagues.
- **Gender:** Give equal opportunities to all learners (girls and boys) to present their findings. Encourage them to participate actively in all learning and teaching activities from the beginning to the end of the teaching and learning process.

2.4. Guidance on introductory activity 2

- In small groups, Invite students to work on Introductory Activity 2 to understand the concept of polynomials, linear equations and algebraic expressions

- Provide clear guidance and directions to perform the activity well.
- Give students time to analyze and discuss the introductory activity.
- Walk around in groups to give advice and appropriate assistance where needed.
- Based on students' experience, prior knowledge, and skills demonstrated in answering the questions for this activity, use different questions to help them make a good presentation that will spark their curiosity about what is in this unit.
- After presenting their findings, harmonize and guide class discussions.
- Point out to students that in the given activity, they may get different answers depending on the sentence they are looking at.
- Try to stimulate students' curiosity about the content of this first unit.

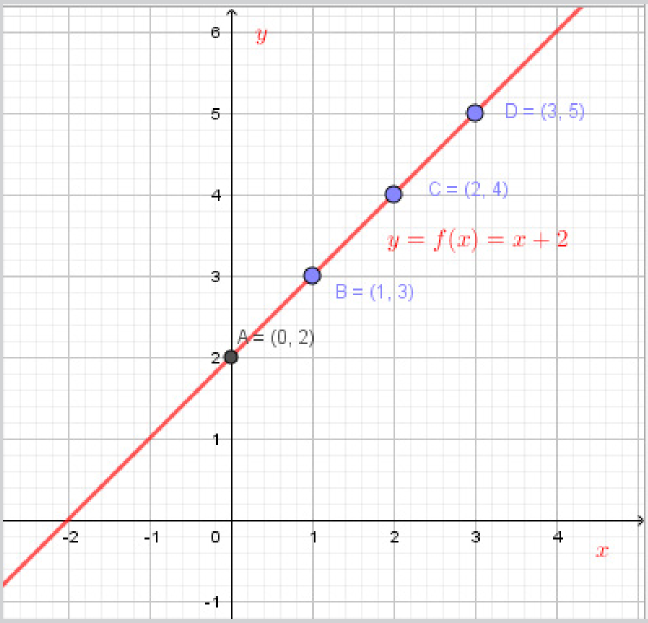
Answer for introductory activity 2

1. If x is the number of pens for a learner, the teacher decides to give him/her two more pens. A learner with one pen will have $(1+2)\,\text{pens} = 3\,\text{pens}$

   a) $y = f(x) = x + 2$

x	-2	-1	0	1	2	3	4
$y = f(x) = x+2$	0	1	2	3	4	5	6
(x,y)	(-2;0)	(-1;1)	(0,2)	(1;3)	(2;4)	(3;5)	(4;6)

   b) The graph obtained is the following:



   c) The graph obtained is a straight line.

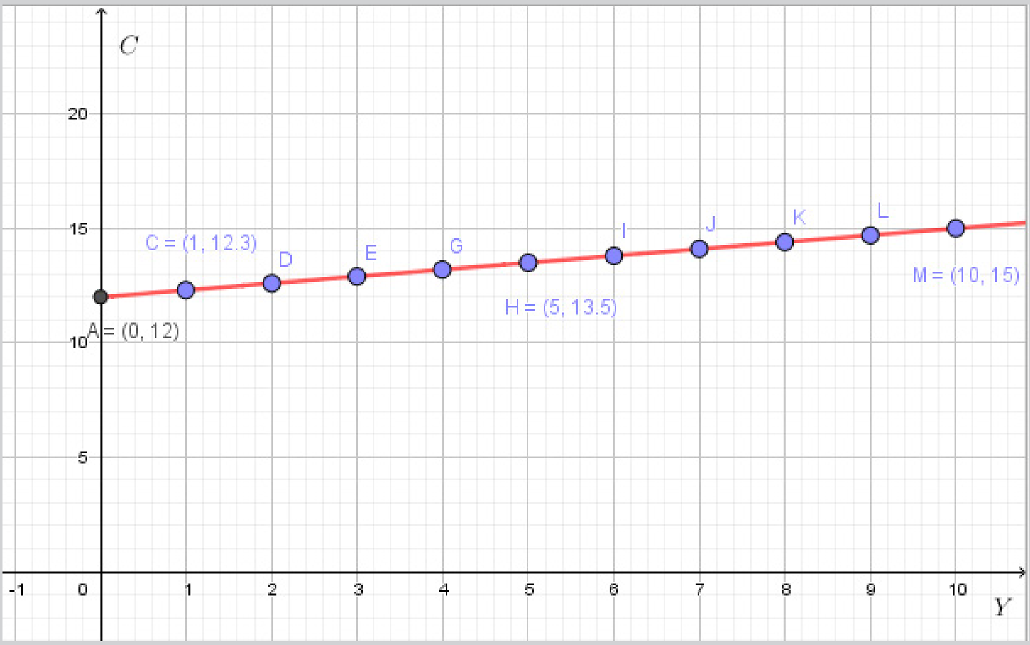
d) $y = x + 2$ this is a linear equation because its graph is a line. Identically, $x + 2 \geq 0$ is a linear inequality.

2. Suppose that average weekly household expenditure on food C depends on average net household weekly income Y according to the relationship $C = 12 + 0.3Y$.

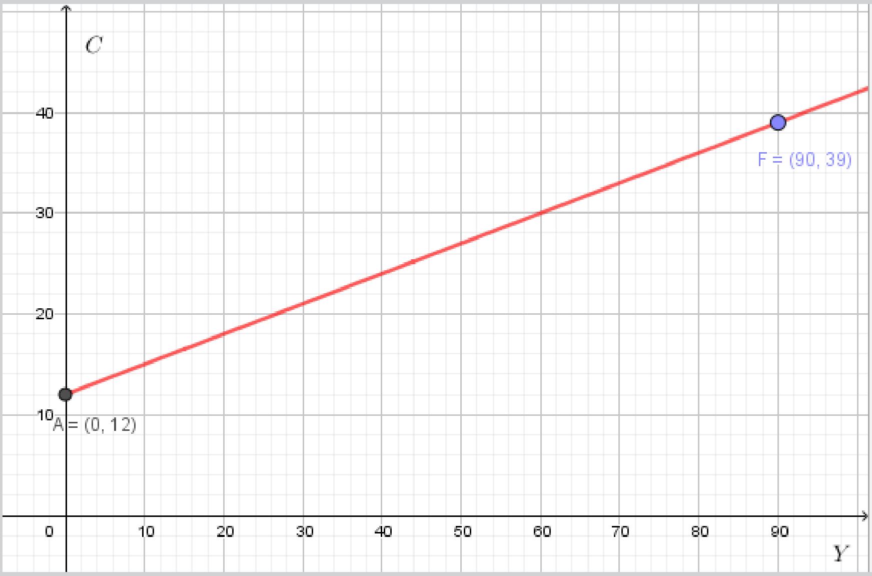
a) For every value of Y, C is a real number. This means that $\forall y \in \mathbb{R}, C \in \mathbb{R}$. The domain of $C(Y)$ is $\mathbb{R}$.

b) Table of value from Y= 0 to Y= 10 and use it to draw the graph of $C = 12 + 0.3Y$

Y	0	1	2	3	4	5	6	7	8	9	10
$C(Y)$	12	12.3	12.6	12.9	13.2	13.5	13.8	14.1	14.4	14.7	15



c) If Y=90, the value of C is $C(90) = 12 + 0.3(90) = 39$



2.5. List of lessons

Headings	**#**	**Lesson title/ sub-headings**	**Learning objectives**	**Number of periods**
2.1 Numerical functions	0	Introductory activity	To arouse the curiosity of student-teacher on the content of unit 2.	1
	1	Generalities on numerical functions	Appreciate the importance of linear functions with their interpretation and analysis	2
	2	Types of functions	Appreciate the types of numerical functions	3
	3	Domain of definition for numerical functions	Determine the domains of definition of different numerical functions.	1
	4	Graph of linear and quadratic functions	Sketch the graph of linear and quadratic functions	1
	5	Parity of numerical function (odd or even)	Differentiate even functions from odd functions.	1

2.2 Equations and inequalities	1	Linear equations	Solve graphically and algebraically linear equations	2
	2	Quadratic equations	Solve algebraically quadratic equations	2
	3	Linear inequalities	Solve graphically and algebraically inequalities.	1
	4	Inequalities products / quotients	Apply inequalities to solve economics and finance related problems	1
	5	Quadratic inequalities	Solve the given quadratic inequalities	1
2.3 Application of linear and quadratic functions in production, finance and economics (Graphical representation and interpretation)	1	Cost function	Perform accurate calculations in algebra in order to interpret and analyse cost function	2
	2	Revenue function	Perform accurate calculations in algebra in order to interpret and analyse revenue function	2
	3	Profit function	Perform accurate calculations in algebra in order to interpret and analyse profit function	2
	4	Demand function	Perform accurate calculations in algebra in order to interpret and analyse demand function	2
	5	Supply function	Perform accurate calculations in algebra in order to interpret and analyse supply function	2
2.4. End unit assessment				1

Lesson 1: Generalities on numerical functions

a) Learning objective:

Appreciate the importance of linear functions with their interpretation and analysis.

b) Teaching resources:

Ruler, T-square, Graph papers, manila papers, digital technology including calculators, interactive multimedia activities, students' book and other Reference textbooks to facilitate research.

Note: If possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or Graph-Calculator and internet can be used.

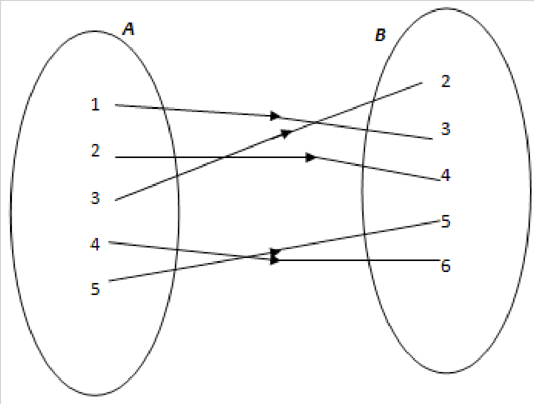
c) Prerequisites/Revision/Introduction:

- Students will learn better this lesson if they have a good understanding on concepts of function and equation studied in ordinary level, and
- Students will learn this lesson better if they have a good knowledge of using the mathematical language, vocabulary and appropriate symbols learned at the ordinary level, a good knowledge of numerical calculations and the interpretation of simple diagrams.

d) Learning activities

- Invite students to form small groups;
- Provide clear instructions and introduce the activity 2.1.1. by guiding the students
- In small groups, ask the students to do activity 2.1.1 from the student book
- Move around to ensure that all students are actively participating in groups
- After a certain amount of time, randomly ask a few groups to present their findings to the whole class;
- Harmonize students' results and make them easier for them to summarize the concepts related to the lesson.
- Use different probing questions and guide students to explore examples and the content given in the student's book to enhance skills on generalities on numerical functions.
- After this step, guide students to do the application activity 2.1.1. then asses their competences and evaluate whether lesson objectives were achieved.

Answers for Activity 2.1.1.



a) The set of elements of A which have images in B is $\{1;2;3;4;5\}$

b) The set of elements in B which have antecedent in A is $\{2;3;4;5;6\}$

c) Each element of the set A has one image in the set B.

e) Answers for application activity 2.1.1

$a) f(2) = 8$

$b) f(-2) = 0$

$c) f(d) = 2d + 8$

Lesson 2: Types of functions

a) Learning objective:

Appreciate the types of numerical functions

b) Teaching resources:

Ruler, T-square, Graph papers, manila papers, digital technology including calculators, interactive multimedia activities, students' book and other Reference textbooks to facilitate research.

Note: If possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or Graph-Calculator and internet can be used.

c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good understanding on concepts of functions studied in ordinary level, and

d) Learning activities

- Invite students to work in group on the activity 2.1.2. found in their student's book;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite group representative from each group to present their findings;
- As a teacher, harmonize the findings from presentation and guide them to explain why they take such type of function.
- Use different probing questions and guide them to explore the content and examples given in the student's book.
- Guide students to be able to differentiate different types of functions: Constant function, Identity, Monomial, Polynomial, Rational and Irrational functions.
- Guide students to classify polynomial functions either by number of terms or by degrees and guide them to establish the general form of a polynomial function as $f(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+...+a_2x^2+a_1x^1+a_0x^0$
- After this step, guide the students to do the application activity 2.1.2. and evaluate whether the learning objectives have been achieved.

Answers for activity 2.1.2.

Constant function	Linear function	Quadratic function	Rational function	Irrational function
$f(x)=2$	$f(x)=2x+1$	$f(x)=x^2$	$h(x)=\dfrac{x^3+2x+1}{x-4}$	$f(x)=\sqrt{x^2+x-2}$
The value of the function is a constant 2	The function is of the first degree and a linear line	The function is of the second degree and a parabola	The function is in the form of fraction or ratios of two polynomials.	The function is in the form of radical or surd.

e) Answers for application activity 2.1.2.

1. $f(x) = x^3 + 2x^2 - 2$ is a polynomial function.
2. $f(x) = \sqrt{x-1}$ is an irrational function
3. $h(x) = \dfrac{x^3 + 2x^2 - 2}{x-5}$ is a rational function.

Lesson 3: Domain of definition for numerical functions

a) Learning objective:

Determine the domain of definition of different numerical functions.

b) Teaching resources:

Student's book and other reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, graphing software such as Geogebra, etc.

c) Prerequisites/Revision/Introduction:

Students should be able to explain the different types of functions learned in previous lesson.

d) Learning activities

- Invite students to work in group on the activity 2.1.3. found in their student's book;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite group representative from each group to present their findings;
- As a teacher, harmonize the findings from presentation and guide them to explain why they take such value(s) which the functions are not defined
- Use different probing questions and guide them to explore the content and examples given in the student's book.
- Guide students to be able determine domain of definition for different types of functions
- After this step, guide the students to do the application activity 2.1.3. and evaluate whether the learning objectives have been achieved.

Answers for activity 2.1.3.

1) For all real numbers $f\left(x\right)=x^3+2x+1$ is defined.

2) $f\left(x\right)=\frac{1}{x}$ is not defined for . It means that for $x=0\Rightarrow f\left(0\right)=\frac{1}{0}\notin\mathbb{R}$ (it is impossible to divide by zero in the set of real numbers).

3) $g\left(x\right)=\frac{x+2}{x-1}$ is not defined for . It means that for $x=1,\ f\left(1\right)=\frac{3}{0}\notin\mathbb{R}$.

4) $f(x)=\sqrt{2x+1}$ is not defined for $2x+1<0$. It means all values of $x<-\frac{1}{2}$

e) Answers for application activity 2.1.3.

1.	$f\left(x\right)=\sqrt{4x-8}$	$domf=\left[2,+\infty\right[$
2.	$g\left(x\right)=\sqrt{x^2+5x-6}$	$domg=\left]-\infty,-6\right]\cup\left[1,+\infty\right[$
3.	$f(x)=\frac{x-2}{x^2-25}$	$domf=\left]-\infty,-5\right[\cup\left]-5,5\right[\cup\left[5,+\infty\right[$

Lesson 4: Graph of linear and quadratic functions

a) Learning objective:

Sketch the graph of linear and quadratic functions

b) Teaching resources:

Student's book and other reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, graphing software such as Geogebra, etc.

c) Prerequisites/Revision/Introduction:

- Students should be able to explain the different types of functions learned in previous lesson.
- The students should be able to find out the domain of definition of given numerical functions.

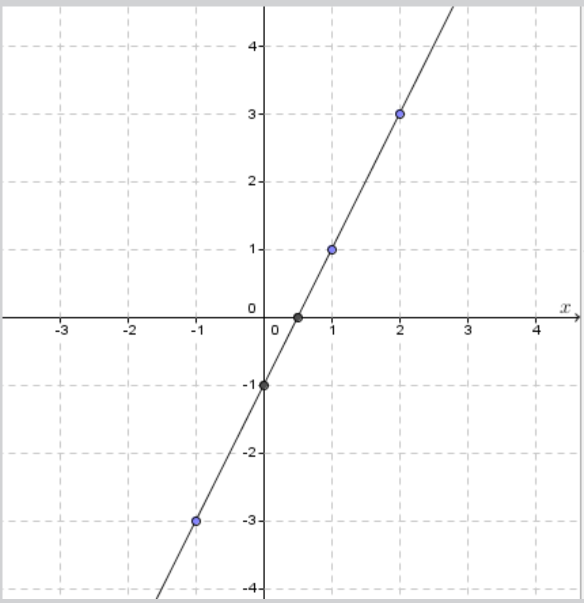
d) Learning activities

- Invite students to work in group on the activity 2.1.4. found in their student's book;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite group representative from each group to present their findings;
- As a teacher, harmonize the findings from presentation and guide them to explain how to sketch the graph of linear and quadratic functions
- Use different probing questions and guide them to explore the content and examples given in the student's book.
- Guide students to be able sketch the graph of linear and quadratic functions
- After this step, guide the students to do the application activity 2.1.4. and evaluate whether the learning objectives have been achieved.

Answers for activity 2.1.4.

1. $y = 2x - 1$ for $-3 \leq x \leq 3$

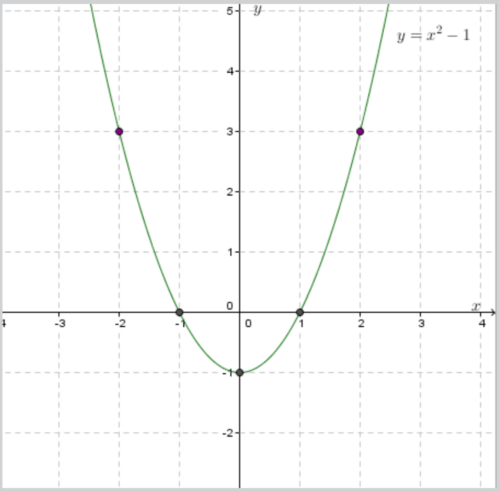
x	-3	-2	-1	0	1	2	3
Y=2x-1	-7	-5	-3	-1	1	3	5



The graph for y=2x-1 is a straight line.

2. $y = x^2 - 1$ for $-3 \le x \le 3$

x	-3	-2	-1	0	1	2	3
$y = x^2 - 1$	8	3	0	-1	0	3	8

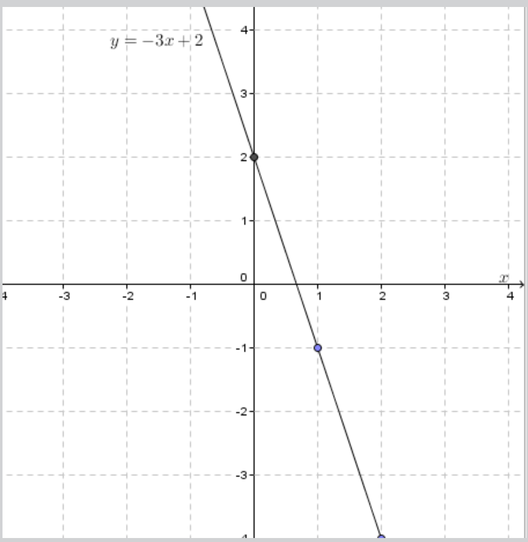


The graph for $y = x^2 - 1$ is a curved line.

e) Answers for application activity 2.1.4.

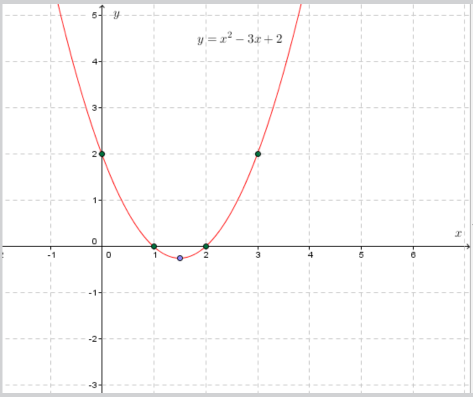
1. a) y=-3x+2

x	-1	0	1	2
Y=-3x+2	5	-2	-1	-4



b) $y = x^2 - 3x + 2$

x	-1	0	1	2	3
$y = x^2 - 3x + 2$	6	2	0	0	2



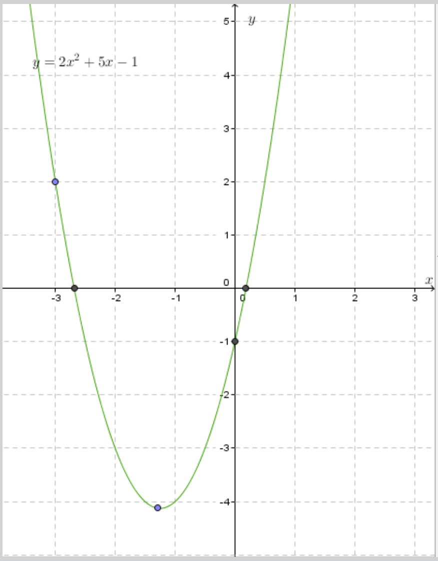
2. a) $y = 2x^2 + 5x - 1$

vertex $v(h,k)$ with $h = -\frac{b}{2a} = -\frac{5}{4}$ *and* $k = 2\times\left(-\frac{5}{4}\right)^2 + 5\times\left(-\frac{5}{4}\right) - 1 = -\frac{33}{8}$

$\Rightarrow v = \left(-\frac{5}{4}, -\frac{33}{8}\right)$ axis of symmetry $x = -\frac{5}{4}$

- if $x = 0 \Rightarrow y = -1$ and y-intercept is $(0,-1)$
- if $y = 0 \Rightarrow x = \frac{-5+\sqrt{33}}{4} \simeq 0.186$ or $x = \frac{-5-\sqrt{33}}{4} \simeq -2.68$

  then x-intercept is $(0.18, 0)$ *or* $(-2.68, 0)$



c) $y = 3x^2 + 8x - 6$

vertex $v(h,k)$ with

$$h = -\frac{b}{2a} = -\frac{8}{6} = -\frac{4}{3} \; and \; \text{k}=3\times\left(-\frac{4}{3}\right)^2 + 8\times\left(-\frac{4}{3}\right) - 6 = -\frac{34}{3}$$

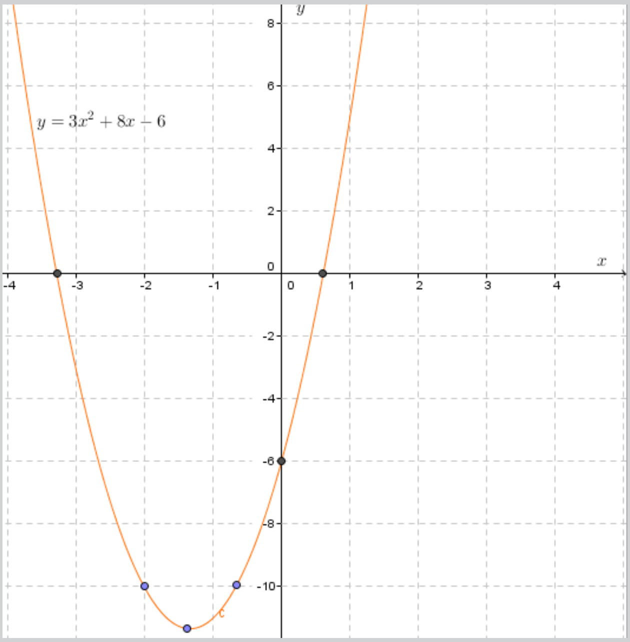
$$\Rightarrow v = \left(-\frac{4}{3}, -\frac{34}{3}\right)$$

axis of symmetry $x = -\frac{4}{3}$

if $x = 0 \Rightarrow y = -6$ and y-intercept is $(0,-6)$

if $y = 0 \Rightarrow x = \frac{-8+\sqrt{136}}{6} \simeq 0.61$ or $x = \frac{-8-\sqrt{136}}{6} \simeq -3.27$

then x-intercept is $(0.61,0)$ *or* $(-3.27,0)$



3) **a)** $\mathbf{C}(10) = \mathbf{80}(\mathbf{10}) + \mathbf{150}; x = 10$

$= \mathbf{800} + \mathbf{150}$

$= \mathbf{950}$

Therefore, the cost of 10 products is 950 FRW.

b) $C(x) = 15,000$

$15,000 = \mathbf{80}x + \mathbf{150}$

$$x = \frac{15,000 - 150}{80}$$

$$x = 185.625$$

Since, the company can produce 185 products for 15 000 FRW.

c) The restriction on the domain x ≥ 0 is necessary because it makes no sense when the number of products produced is negative.

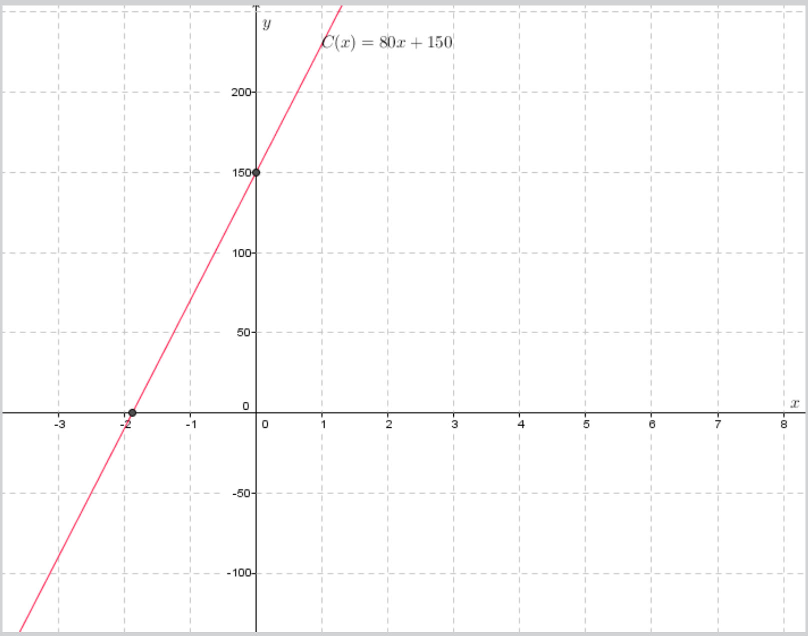
d) $x = 0$

$f(0) = 80(0) + 150 = 150$

It has no meaning to pay zero product. But this amount of 150FRW is considered as the fixed or start-up of the venture.

e) We recognize that the slope is defined by

$$m = \frac{change\ in\ y - coordinates}{change\ in\ x - coordinates} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{950 - 150}{10 - 0} = 80$$



Lesson 5: Parity of a function

a) Learning objective

Differentiate even functions from odd functions.

b) Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, gridded paper, and graphing software such as Geogebra, etc…

c) Prerequisites/Revision/Introduction

Students will learn this lesson better if they are well acquainted with the content of Unit 2 of S2 and the previous content of this unit

d) Learning activities

- Invite students to work in groups and do the activity 2.1.5 found in the student's book;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Verify and identify groups with various working steps;
- Invite one member from each group with various working steps to present their work to the whole class.
- As a teacher, harmonize the findings from presentation and guide them to enhance the characteristics of even functions and odd functions.
- Together with students, use graphs for simple functions to illustrate the characteristics of even and odd functions: The graph of even function is symmetric about the vertical axis (the line $x=0$ is the axis of symmetry) while the graph of odd function looks the same when rotated through half a revolution about 0 (the point $(0,0)$ is the centre of symmetry for its parts).
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to verify the parity of different functions.
- After this step, guide students to do the application activity 2.1.5. and evaluate whether lesson objectives were achieved.

Answer for activity 2.1.5.

1. $f(x)=x^2+3$

- $f(-x)=(-x)^2+3=x^2+3$
- $-f(x)=-(x^2+3)=-x^2-3$

$\therefore f(-x)=f(x) \quad and \quad -f(x)\neq f(-x)$

2. $f(x)=\sqrt[3]{x^2+x}$

$f(-x)=\sqrt[3]{(-x)^2+(-x)}=\sqrt[3]{x^2-x}$

$-f(x)=-\sqrt[3]{x^2+x}=\sqrt[3]{-x^2-x}$

$\therefore f(-x)\neq -f(x)$

3. $f(x)=\dfrac{x^2-3}{x^2+1}$

$f(-x)=\dfrac{(-x)^2-3}{(-x)^2+1}=\dfrac{x^2-3}{x^2+1}$

$-f(x)=-\dfrac{x^2-3}{x^2+1}=\dfrac{-x^2+3}{x^2+1}$

$\therefore -f(x)\neq f(-x)$

e) Answers for application activity 2.1.5

1) $f(x)=2x^2+2x-3$

- $f(-x)=2(-x)^2+2(-x)-3=2x^2-2x-3$
- $-f(x)=-(2x^2+2x-3)=-2x^2-2x+3$

$f(-x)\neq -f(x) and$

$f(-x)\neq -f(x)$

$\therefore f(x)=2x^2+2x-3$ is neither odd nor even.

2) $h(x)=\dfrac{x^2+4}{x^2-4}$ is even function.

Lesson 6: Linear equations

a) Learning objective:

Solve graphically and algebraically linear equations.

b) Teaching resources:

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, gridded paper, and graphing software such as Geogebra, etc…

c) Prerequisites/Revision/Introduction:

Students will do well in this unit if they have sufficient experience with equations acquired from ordinary level and with linear functions learned in previous lessons.

d) Learning activities

- Invite students to work in group and answer question 1 of activity 2.2.1.
- Ask each group to share their answers with another group, and ask them to support each other if they have greater challenges in solving this task.
- Ask the group representative to share their findings with the whole class during a class discussion;
- As a teacher, harmonize the work done on question 1 of activity 2.2.1. through presentation and insisting on on: increasing function, value of a function at a point, initial value for a function, solving an equation, the set of solution for equation.
- Use different questions and examples from student book and guide students on how to solve linear equations.
- Let students go through the application activity 2.2.1. to answer question 1 of this activity and evaluate whether the objectives of the lesson were achieved.

Answers for activity 2.2.1.

1) $x+1=5$ $x+1-1=5-1$ $x=4$	2) $2x-4=0$ $2x-4+4=0+4$ $2x=4$ $\frac{2x}{2}=\frac{4}{2}$ $x=2$	3) $2x+1=-5$ $x+1-1=-5-1$ $x=-6$	4) $x-4=10$ $x-4+4=10+4$ $x=14$

e) Answers for application activity 2.2.1.

1. $x+5=9 \quad\Rightarrow\quad x+5-5=9-5 \quad\Rightarrow\quad x=4$
2. $6x+5=5 \quad\Rightarrow\quad 6x+5-5=5-5 \quad\Rightarrow\quad 6x=0 \quad\Rightarrow\quad x=0$
3. $x-2=3 \quad\Rightarrow\quad x-2+2=3+2 \quad\Rightarrow x=5$
4. $25=2x-5 \quad\Rightarrow\quad 30=2x \quad\Rightarrow x=15$
5. $-5=x-1 \quad\Rightarrow\quad -5+1=x \quad\Rightarrow x=-4$
6. $3x-4=2x+1 \quad\Rightarrow\quad x=5$
7. $x+5=9x+1 \quad\Rightarrow\quad -8x=-4 \quad\Rightarrow\quad x=\frac{1}{2}$
8. $-6x-5=9 \quad\Rightarrow\quad -6x=14 \quad\Rightarrow\quad x=-\frac{14}{6}=-\frac{7}{3}$
9. $x+100=99 \quad\Rightarrow x=99-100 \quad\Rightarrow x=-1$
10. $6x-51=9 \quad\Rightarrow 6x=9+51 \quad\Rightarrow 6x=60 \quad\Rightarrow x=10$

Lesson 7: System of linear equations

a) Learning objective:

Solve graphically and algebraically system of linear equations

b) Teaching resources:

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, gridded paper, and graphing software such as Geogebra, etc…

c) Prerequisites/Revision/Introduction:

Students will do well in this unit if they have sufficient experience with system of equations acquired from ordinary level and with linear functions learned in previous lessons.

d) Learning activities

- Invite students to work in group and answer question 2 of activity 2.2.1.
- Ask each group to share their answers with another group, and ask them to support each other if they have greater challenges in solving this task.
- Ask the group representative to share their findings with the whole class during a class discussion;

- As a teacher, harmonize the work done on question 2 of activity 2.2.1. through presentation and insisting on on: increasing function, value of a function at a point, initial value for a function, solving an equation, the set of solution for equation.
- Use different questions and examples from student book and guide students on how to solve linear equations.
- Let students go through the application activity 2.2.1. to answer question 2 of this activity and evaluate whether the objectives of the lesson were achieved.

Answers for activity 2.2.1. question 2

Given the two linear equations in 2 unknowns $x-y=1$ and $x+y=1$

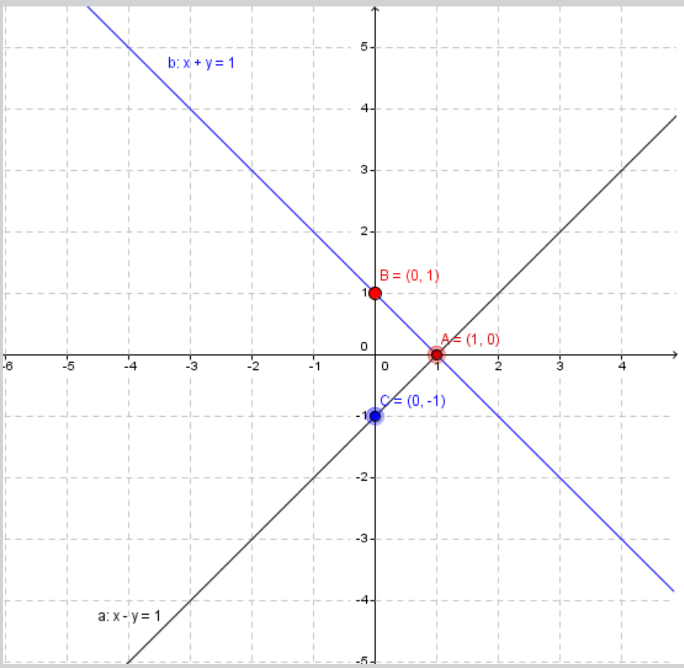
a. If $x=1$, the value of y in the equation $x-y=1$ is 0 and the point (x,y) is $(1,0)$

   If $x=1$, the value of y in the equation $x+y=1$ is 0 and the point (x,y) is $(1,0)$

   If $x=0$, the value of y in the equation $x-y=1$ is -1 and the point (x,y) is $(0,-1)$

   If $x=0$, the value of y in the equation $x+y=1$ is -1 and the point (x,y) is $(0,1)$

b. Graph of $x-y=1$ and $x+y=1$.



c. The point of intersection for the two lines is the point A (1,0)

e) Answers for application activity 2.2.1. question 2

a. $\begin{cases} 4y+x=8 \\ -x+y=2 \end{cases}$

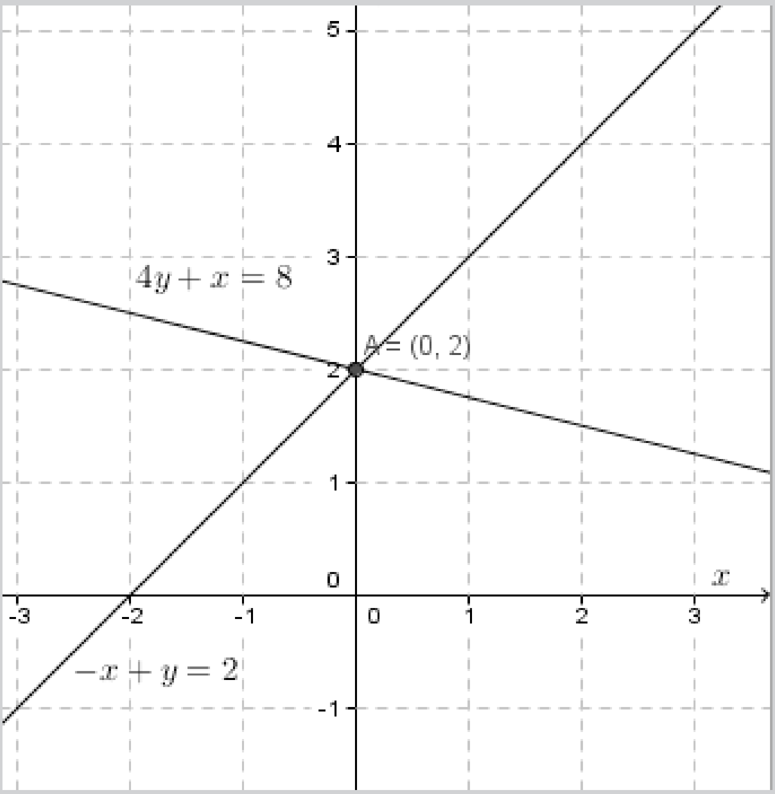
- From the equation (1) , $4y+x=8 \Rightarrow x=8-4y$
- From the equation (2), $-x+y=2 \Rightarrow x=y-2$
- Equalize the values of x from equation (1) and (2)

$8-4y=y-2$

$y=2$

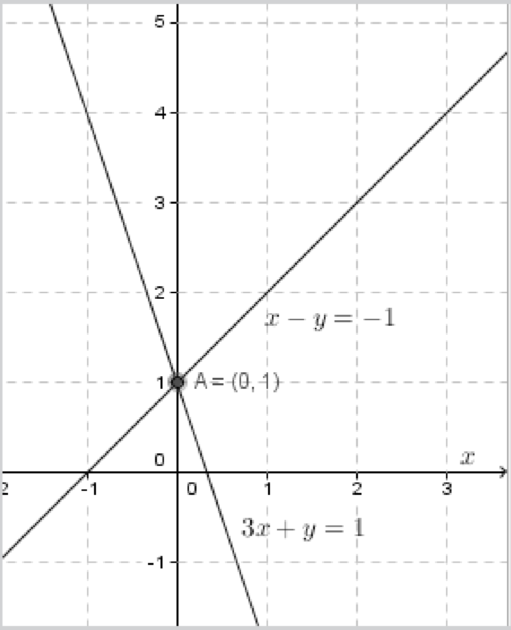
- As $x=y-2$, substituting the value of , we find the value of $x=0$
- The solution set is $S=\{(0,2)\}$

Graph of the two linear equations



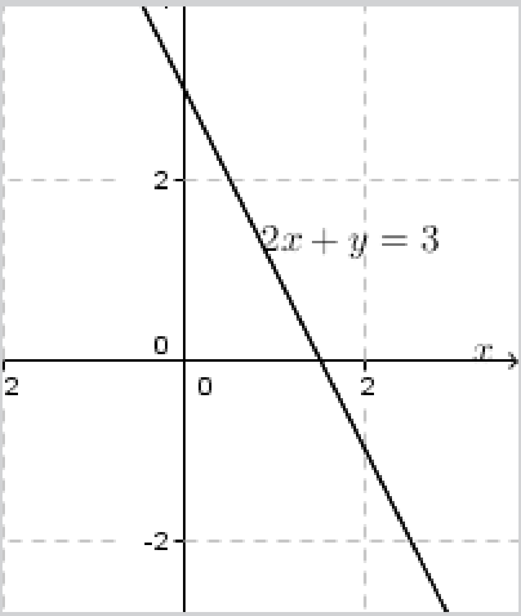
Solution of the system is $S=\{(0,2)\}$

b. $\begin{cases} 3x+y=1 \\ x-y=-1 \end{cases}$



The solution $S=\{(0,1)\}$

c. $\begin{cases} 2x+y=3 \\ 6x+3y=9 \end{cases}$



Since the two lines are coinciding the system has infinite solutions. The solution is made of the entire line. $S=\{(x,y)\in\mathbb{R}^2 : 2x+y=3\}$.

Lesson 8: Quadratic equations

a) Learning objective:

Solve algebraically quadratic equations.

b) Teaching resources:

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, gridded paper, and graphing software such as Geogebra, etc…

c) Prerequisites/Revision/Introduction:

Students will do well in this unit if they have sufficient experience with equations acquired from ordinary level and with linear equation learned in previous lessons.

d) Learning activities

- Invite students to join small groups
- Invite students to work in their respective groups on the questions for Activity 2.2.2.
- Move in all groups guiding struggling students to work on Activity 2.2.2.
- Ask each group to share their answers with another group, and ask them to support each other if they have greater challenges in solving this task.
- Ask the group representative to share their findings with the whole class during a class discussion;
- As a teacher, harmonize the work done through presentation and insisting on on: factorizing, using the formula and completing the squares
- Use different questions and examples from student book and guide students on how to solve quadratic equations.
- Let students go through the application activity 2.2.2. and evaluate whether the objectives of the lesson were achieved.

Answers for activity 2.2.2.

1) $x^2+2x-24=0$

$Sum=2$ for $-4;6$

$\text{Product}=-24$

$$x^2+2x-24=0$$
$$x^2-4x+6x-24=0$$
$$\left(x^2-4x\right)+\left(6x-24\right)=0$$
$$x\left(x-4\right)+6\left(x-4\right)=0$$
$$(x-4)(x+6)=0$$

$x=4$ and $x=-6$

2) If the length is twice the width and suppose that width is x, then;

$W=x$

$L=2x$

For $A=L\times W$

$900=2x\times x$

$900=2x^2$

$x=\pm\sqrt{\frac{900}{2}}$

$x=\pm\sqrt{450}$

$x=\pm\sqrt{225\times 2}=\pm 15\sqrt{2}$

$\therefore x=+15\sqrt{2}=W; L=15\sqrt{2}\times 2=30\sqrt{2}$

e) Answers for application activity 2.2.2.

A. $1)S=\{-2,-4\}$

$2)S=\{3,-1\}$

B. $1)S=\{11,1\}$

$2)S=\{-7,5\}$

C. $1)S=\{-8,3\}$

$2)S=\{9,4\}$

D. The price of the demand drops to 1000 units:

$$D=2000+100P-6P^2 \Rightarrow 1000=2000+100P-6P^2$$
$$-6P^2+100P+1000=0 \Rightarrow \Delta=10000+24000=34000$$
$$\sqrt{\Delta}=20\sqrt{85}$$
$$P_1=\frac{-100+20\sqrt{85}}{-12}=-7.03$$
$$P_2=\frac{-100-20\sqrt{85}}{-12}=23.69$$

As the price can't be negative, P_1 is rejected and we consider only $P_2=23.69$

Lesson 9: Linear inequalities

a) Learning objective:

Solve graphically and algebraically inequalities

b) Teaching resources:

Graph papers, manila papers, calculators, markers, pens, graph editors such as Geogebra (where possible).

c) Prerequisites/Revision/Introduction:

Students will do well in this unit if they have sufficient experience with linear inequalities in one unknown, explored in ordinary level and previous lesson functions and equations.

d) Learning activities

- Invite students to sit in pairs
- Invite students to work in pairs on the questions for Activity 2.2.3
- Move around to support and guide struggling students in their work
- Ask each pair of students to share their answers with another pair of students, and ask them to support each other when they have greater challenges in solving this activity.
- Ask the group representative to share their findings with the whole class during a class discussion;
- As a teacher, harmonize the work done through presentation and insisting on how to write and present solutions of inequalities
- Use different questions and examples from student book and guide students on how to solve inequalities
- Let students go through the application activity 2.2.3. and evaluate whether the objectives of the lesson were achieved.

Answers for activity 2.2.3.

1) $x < 5$ represents all numbers less than 5, for example 4, 3, 2, 1, 0, -1, -2, etc...
2) $x > 0$ represents all numbers greater than 0, for example 1, 2, 3, 4, 5, 6 etc…
3) $-4 < x < 12$ includes for example -3,-2,-1,0,1,2,3,...,11.

e) Answers for application activity 2.2.3

1) $S = \left]-\infty, 9\right[$

2) $S = \left]-\infty, 10\right[$

3) $S = \left]-\infty, 5\right]$

4) $S = \left[\dfrac{26}{3}, +\infty\right[$

Lesson 10: Inequalities: products / quotients

a) Learning objective:

Apply inequalities to solve economics and finance related problems

b) Teaching resources:

Graph papers, manila papers, markers, calculators, markers, pens, graph editors such as Geogebra (where possible).

c) Prerequisites/Revision/Introduction:

Students will do well in this unit if they have sufficient experience with linear inequalities in one unknown, explored in ordinary level, basic concept of algebra, and previous lesson on functions and equations.

d) Learning activities

- Invite students to sit in small groups
- Invite students to work in small groups on the questions for Activity 2.2.4
- Move around to support and guide struggling students in their work
- Ask each group to share their answers with another group, and ask them to support each other when they have greater challenges in solving this activity 2.2.4.
- Ask the group representative to share their findings with the whole class during a class discussion;
- As a teacher, harmonize different answers given by students on the activity 2.2.4.
- Use different questions and examples from student book and guide them on how to solve inequalities in the set of real numbers
- Let students go through the application activity 2.2.4. and evaluate whether the objectives of the lesson were achieved.

Answers for activity 2.2.4.

Solve inequalities in the set of real numbers

1. Start by solving $(x+1)(x-1)=0$

   $x+1=0$ or $x-1=0$

   $\Rightarrow x=-1$ $\Rightarrow x=1$

   The next is to find the sign table.

x	$-\infty$		-1		1		$+\infty$
$x+1$		-	0	+		+	
$x-1$		-		-	0	+	
$(x+1)(x-1)$		+	0	-	0	+	

Since the inequality is $(x+1)(x-1)<0$; we will take the interval where the product is negative. Thus, $S=]-1,1[$

2. Start by solving

$\frac{x+2}{x-1}=0; x-1\neq 0$

$x+2=0$ or $x-1=0$

$\Rightarrow x=-2$ $\Rightarrow x=1$

The next is to find the sign table.

x	$-\infty$		-2		1		$+\infty$
$x+2$		-	0	+		+	
$x-1$		-		-	0	+	
$\frac{x+2}{x-1}$		+	//	-	0	+	

Since the inequality is $\frac{x+2}{x-1}\leq 0$; we will take the interval where the quotient is negative. Thus, $S=]-2,1]$.

e) Answers for application activity 2.2.4.

1. $S = \left]-\infty, -3\right[\cup \left[+3, +\infty\right[$

2. $\left]-\infty, -2\right[\cup \left[+3, +\infty\right[$

3. $x^2 - 10x - 20 > 0 \Rightarrow x_1 = 5 + 3\sqrt{5}; x_2 = 5 - 3\sqrt{5}$

$(x - 5 - 3\sqrt{5})(x - 5 + 3\sqrt{5}) = 0$

x	$-\infty$ $\quad 5-3\sqrt{5} \quad 5+3\sqrt{5} \quad$ $+\infty$
$(x-5+3\sqrt{5})$	- - 0 + + + + +
$(x-5-3\sqrt{5})$	- - - - 0 + + + +
$(x-5+3\sqrt{5})(x-5-3\sqrt{5})$	+ + + 0 - - - 0 + + +

$s = \left]-\infty, 5-3\sqrt{5}\right[\cup \left]5+3\sqrt{5}, +\infty\right[$

4) $6x^2 - 5x + 1 < 0 \Rightarrow x_1 = \frac{1}{2}; x_2 = \frac{1}{3}$

x	$-\infty$ $\quad +\frac{1}{2} \quad +\frac{1}{3} \quad$ $+\infty$
$(x-\frac{1}{2})$	- - - 0 + + + + + +
$(x-\frac{1}{3})$	- - - - - - 0 + + + +
$(x-\frac{1}{2})(x-\frac{1}{3})$	+ + + 0 - - - 0 + + + +

$s = \left]\frac{1}{3}, \frac{1}{2}\right[$

5) $x^2 + 2x + 12 > 0$; the solution set is the set of real $\mathbb{R}$, because $\Delta < 0$ same as question

Lesson 12: Cost function

a) Learning objective:

Perform accurate calculations in algebra in order to interpret and analyse cost function

b) Teaching resources:

Manila papers, markers, rulers, graph papers, calculators and where possible use the mathematics software for graphing such as Geogebra.

c) Prerequisites/Revision/Introduction:

Students will perform well in this unit if they learnt well the content of previous lessons in this unit and basic concepts of algebra from unit 1.

d) Learning activities

- Invite students to sit in small groups and give them instructions to the activity 2.3.
- Invite students to work in group and do the activity 2.3. question 1 and question 2.b found in student's book.
- Move around the class to moderate where needed and provide more clarity about any challenges they may encounter during their work; Identify groups with different work steps.
- Invite each group with different working steps to present their answers in a whole class discussion;
- As a teacher, harmonize the insights from the students' presentation and highlight Variable costs and Fixed costs
- Encourage students to investigate the content and examples in 2.3.1. in the student's book.
- After this step, invite students to do application activity 2.3. question 1 and question 2.i. and evaluate whether the learning objectives have been achieved.

A. 1) $S=\{-2,-4\}$

2) $S=\{3,-1\}$

B. 1) $S=\{11,1\}$

2) $S=\{-7,5\}$

C. 1) $S=\{-8,3\}$

2) $S=\{9,4\}$

D. The price of the demand drops to 1000 units:

$$D=2000+100P-6P^2 \Rightarrow 1000=2000+100P-6P^2$$

$$-6P^2+100P+1000=0 \Rightarrow \Delta=10000+24000=34000$$

$$\sqrt{\Delta}=20\sqrt{85}$$

$$P_1=\frac{-100+20\sqrt{85}}{-12}=-7.03$$

$$P_2=\frac{-100-20\sqrt{85}}{-12}=23.69$$

As the price can't be negative, P_1 is rejected and we consider only $P_2=23.69$

Note: this function will change for Revenue, demand and supply function due to inputs

Lesson 13: Revenue function

a) Learning objective:

Perform accurate calculations in algebra in order to interpret and analyse revenue function

b) Teaching resources:

Manila papers, markers, rulers, graph papers, calculators and where possible use the mathematics software for graphing such as Geogebra.

c) Prerequisites/Revision/Introduction:

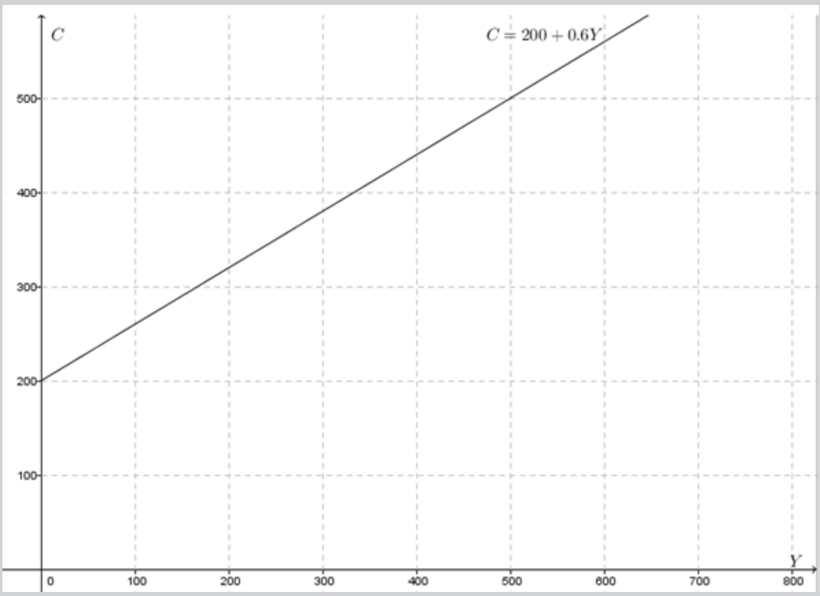
Students will perform well in this unit if they learnt well the content of previous lessons in this unit and basic concepts of algebra from unit 1.

d) Learning activities

- Read the question 1 of activity 2.3 and vary the question to make a product of consumer spending and income, then function become $R(Y)=120Y$
- Ask students to draw the function found in activity 2.3 and new function $R(Y)=120Y$.
- Guide students to discover that $R(Y)=120Y$ is a revenue function
- Invite students to work in group and do the activity 2.3. question 2.ii found in student's book.
- Move around the class to moderate where needed and provide more clarity about any challenges they may encounter during their work; Identify groups with different work steps.
- Invite each group with different working steps to present their answers in a whole class discussion;
- As a teacher, harmonize the insights from the students' presentation and highlight that **Revenue** = **Price x Quantity**, *so* **R** = pq.
- Encourage students to investigate the content and examples in 2.3.2. in the student's book.
- After this step, invite students to do application activity 2.3. question 2. ii. and evaluate whether the learning objectives have been achieved.

Answers for activity 2.3, question 1

Graph of the revenue function $C=200+0.6Y$



Note: this function will change for Revenue, demand and supply function due to inputs

Lesson 14: Profit function

a) Learning objective:

Perform accurate calculations in algebra in order to interpret and analyse profit function

b) Teaching resources:

Manila papers, markers, rulers, graph papers, calculators and where possible use the mathematics software for graphing such as Geogebra.

c) Prerequisites/Revision/Introduction:

Students will perform well in this unit if they learnt well the content of previous lessons in this unit and basic concepts of algebra from unit 1.

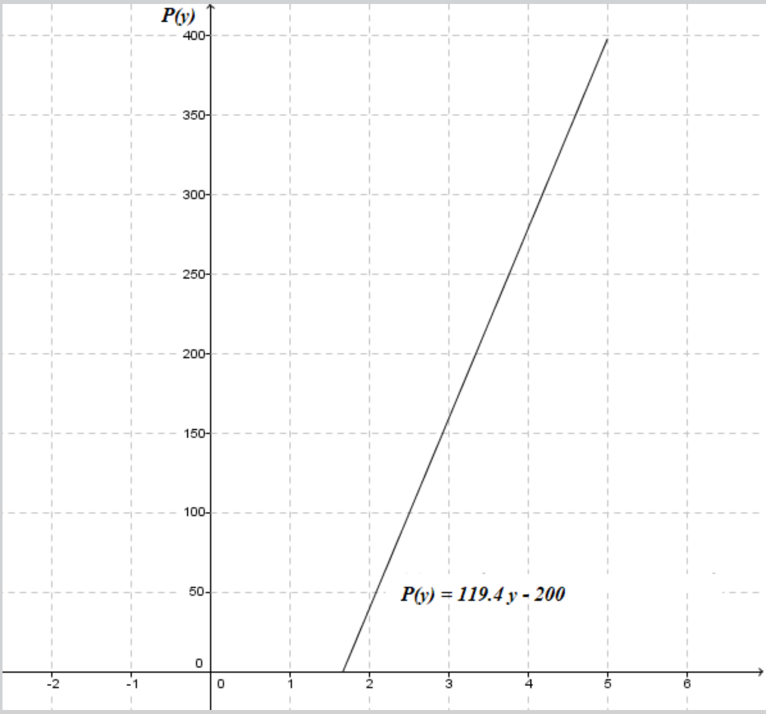
d) Learning activities

- Read the question 1 of activity 2.3 and vary the question to find out profit by taking Revenue Minus Cost given by $P(Y)=120Y-(200+0.6Y)$
- Ask students to draw the new profit function $P(Y)=119.4Y-200$
- Guide students to discover that $P(Y)=119.4Y-200$ is a profit function
- Invite them to work in groups and do the activity 2.3. question 2.C found in student's book.
- Move around the class to moderate where needed and provide more clarity about any challenges they may encounter during their work; Identify groups with different work steps.
- Invite each group with different working steps to present their answers in a whole class discussion;
- As a teacher, harmonize the insights from the students' presentation and highlight that:

  Profit = Revenue – Cost or $P(x)= R(x) - C(x)$where $P(x)$if profit function, $R(x)$ is revenue function and $C(x)$ is cost function
- Encourage students to investigate the content and examples in 2.3.3. in the student's book.
- After this step, invite students to do application activity 2.3. question 2. iii. and evaluate whether the learning objectives have been achieved.

Answers for activity 2.3, question 1

Graph of profit function $P(Y) = 119.4Y - 200$



NB: During graph interpretation, the teacher helps students analyze only that part of a graph in the 1st quadrant where the meaning of the profit function is evidently observed.

Question 2.iii The profit a business makes is equal to the revenue it takes in minus what it spends as costs. To obtain the profit function, subtract costs from revenue.

$$\mathrm{P}(x) = R(x) - C(x) = 300x - 2x^2 - (5000 + 40x)$$

$$= -2x^2 + 260x - 5000$$

Lesson 15: Demand function

a) Learning objective:

Perform accurate calculations in algebra in order to interpret and analyse demand function

b) Teaching resources:

Manila papers, markers, rulers, graph papers, calculators and where possible use the mathematics software for graphing such as Geogebra.

c) Prerequisites/Revision/Introduction:

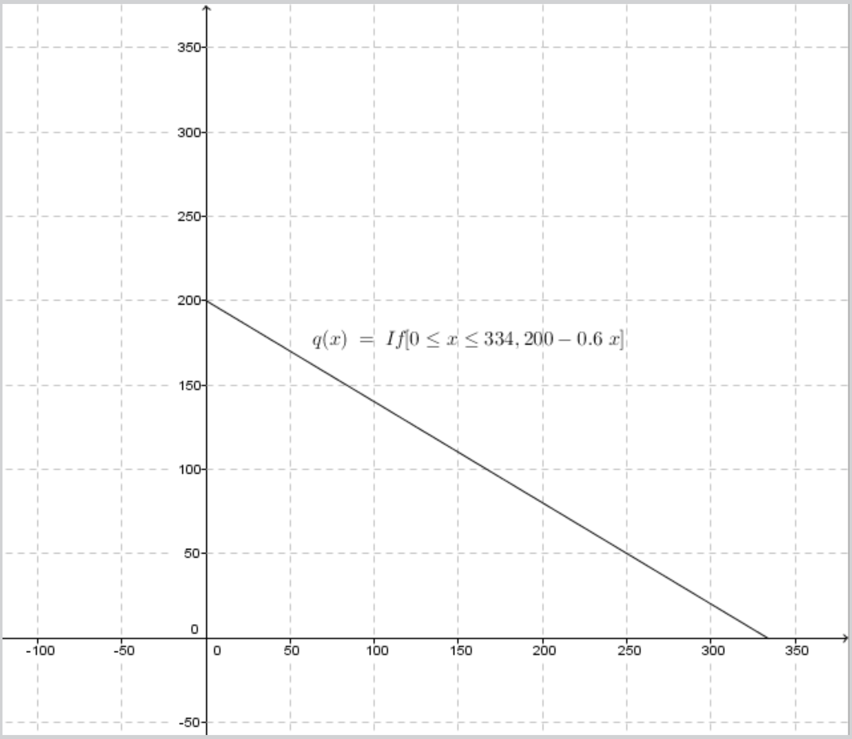
Students will perform well in this unit if they learnt well the content of previous lessons in this unit and basic concepts of algebra from unit 1.

d) Learning activities

- Read the question 1 of activity 2.3 and vary the question to form demand function as $Q(x) = 200 - 0.6x$
- Ask students to draw the new profit function $Q(x) = 200 - 0.6x$
- Individually, guide students to discover that $Q(x) = 200 - 0.6x$ is a demand function
- Move around the class to moderate where needed and provide more clarity about any challenges they may encounter during their work; identify groups with different work steps.
- Invite each group with different working steps to present their answers in a whole class discussion;
- As a teacher, harmonize the insights from the students' presentation and highlight that: The demand function is in the form $P = a - bQ$, where a and b are constants/parameters, P is the price and Q is the quantity demanded.
- Encourage students to investigate the content and examples in 2.3.4. in the student's book.
- After this step, invite students to do application activity 2.3. and evaluate whether the learning objectives have been achieved.

Answers for activity 2.3, question 1

Graph of demand function $Q(x) = 200 - 0.6x$



NB: During graph interpretation, the teacher helps students analyze only that part of a graph in the 1st quadrant where the meaning of the demand function is evidently observed.

Lesson 16: Supply function

a) Learning objective:

Perform accurate calculations in algebra in order to interpret and analyse supply function

b) Teaching resources:

Manila papers, markers, rulers, graph papers, calculators and where possible use the mathematics software for graphing such as Geogebra.

c) Prerequisites/Revision/Introduction:

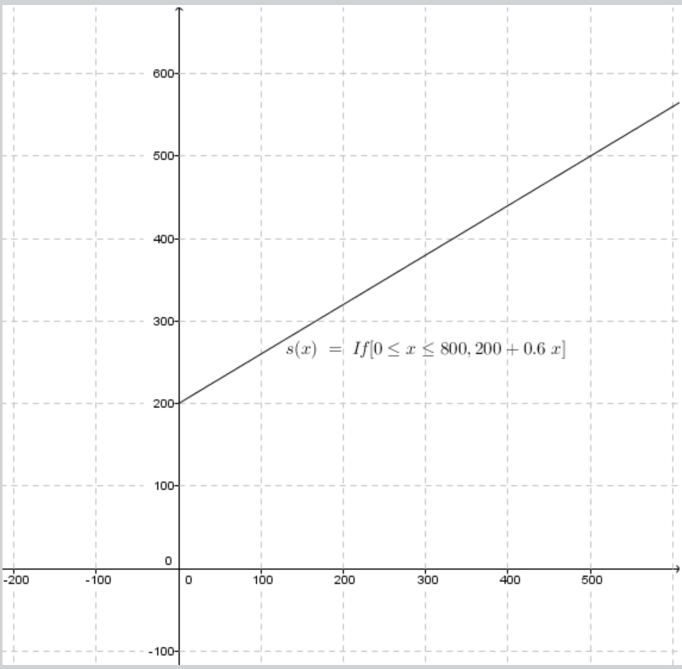
Students will perform well in this unit if they learnt well the content of previous lessons in this unit and basic concepts of algebra from unit 1.

d) Learning activities

- Invite students to sit in pairs
- Invite them to work in pairs on the questions for Activity 2.3. question 1
- Ask students to draw the function $S = 200 + 0.6Y$ and interpret it
- Move around to support and guide struggling students in their work
- Ask each pair of students to share their answers with another pair of students, and ask them to support each other when they have greater challenges in solving this activity.
- Ask the group representative to share their findings with the whole class during a class discussion;
- As a teacher, harmonize the work done through presentation inform students that the graph they found seems to be a supply function following the formula $S = a + bY$ where S depends on income Y
- Encourage students to investigate the content and examples in 2.3.5. in the student's book.
- Ask students to go through the application activity 2.3.5. and evaluate whether the objectives of the lesson were achieved.

Answers for activity 2.3, question 1

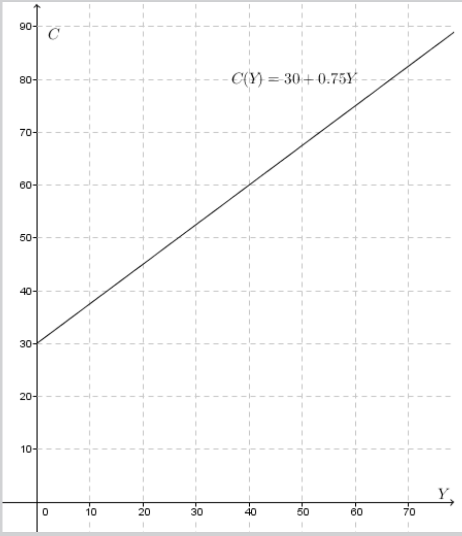
Graph of supply function $S = 200 + 0.6Y$



NB: During graph interpretation, the teacher helps students analyze only that part of a graph in the 1st quadrant where the meaning of the supply function is evidently observed.

e) Answer for application activity 2.3

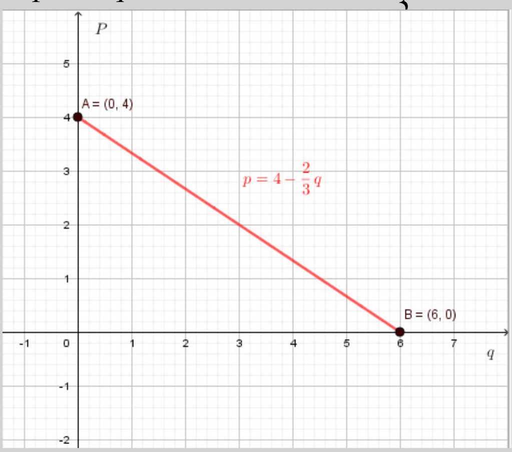
1)



2) a) $f(12) = 60$ means that when the product is 12 units, the price is 60 units of money.

b) Normally the price increases when the quantity is reducing.

3) $75p + 50q = 300$ $\qquad p = 4 - \frac{2}{3}q$



It is clear that the Vertical intercept is $p = 4$ dollars and the horizontal intercept is $q = 6$ units.

4)

a. Cost function = C(x)= 17 500 x+15 000

b. Revenue Function = R((x) = 50 000 x

c. Profit function = Revenue function – Cost function = R(x) - C(x)

= (50 000 x) – (17 500 x+15 000)

= 50 000 x– 17 500 x – 15 000 = 32 500 x -15 000

2.6. Summary of unit 2

Function

A function is a rule that assigns to each element in a set A one and only one element in set B. We can even define a function as any relationship which takes one element of one set and assigns to it one and only one element of second set

Types of functions:

There are different types of functions namely: constant function, Identity, Monomial, Polynomial function, Rational function and irrational function.

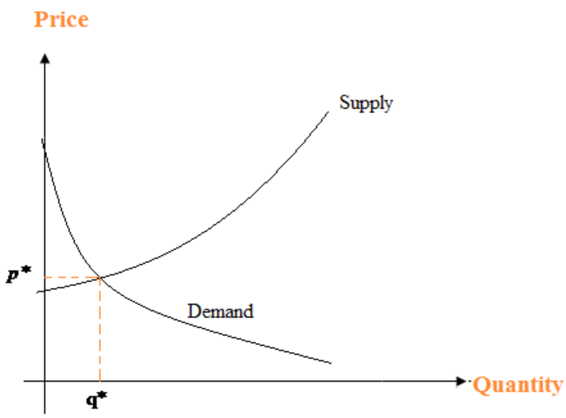
Parity of a function (odd or even

Even function: $f(-x) = f(x)$

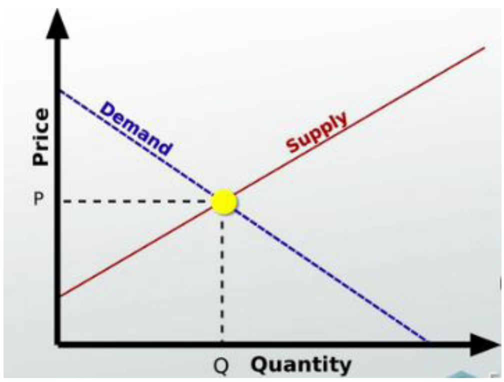
Odd function: $f(-x) = -f(x)$

Equilibrium Price and Quantity

If we plot the supply and demand curves on the same axes, the graphs cross at the equilibrium point. The values p^* and q^* at this point are called the equilibrium price and equilibrium quantity, respectively. It is assumed that the market naturally settles to this equilibrium point.



For quantities which are not linear



For quantities varying linearly

2.7. Additional Information for the teacher

As a teacher, be careful on the following:

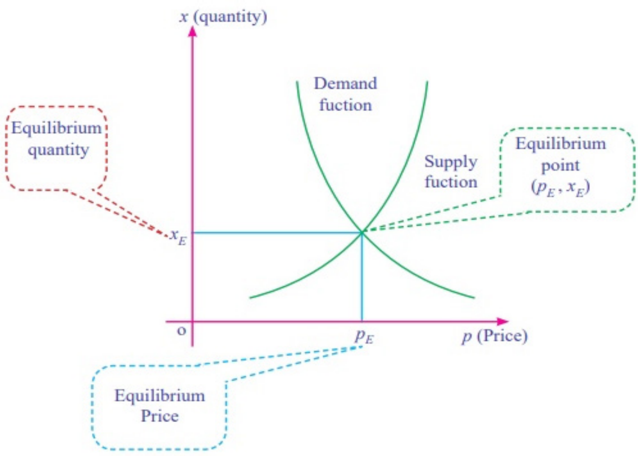
- Emphasize the use of graph paper/gridded paper while students draw the graphs.
- Emphasize and facilitate students' use of geometric materials to improve the quality of graphs.
- Remind students to name axes (x-axis and y-axis).
- Recall them to mention/highlight the origin/intersection point of axes by 0.
- Inform students that when solving quadratic equations , solutions exist or not exist under the following criteria:

  $b^2 - 4a \quad > 0$ Two solutions

  $b^2 - 4a \quad = 0$ One solution

  $b^2 - 4a \quad < 0$ No solution

Use the following graph to explain equilibrium price, equilibrium quantity and equilibrium point



2.8. End unit assessment

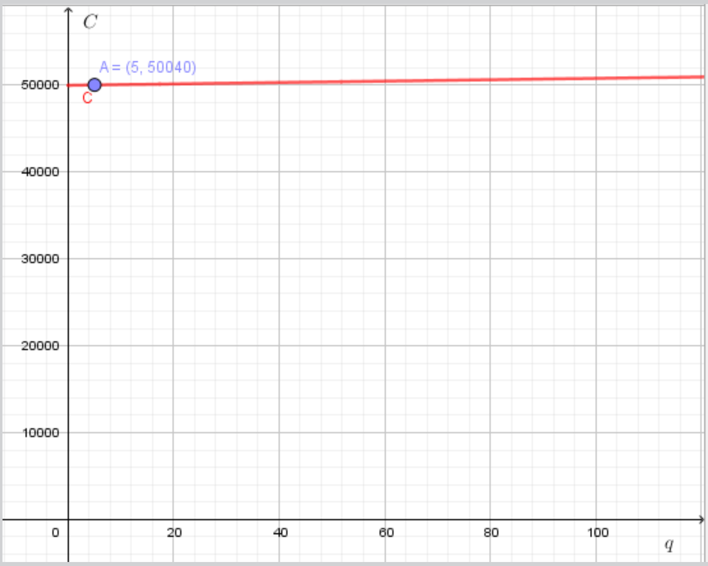
This part provides the answers of end unit assessment activities designed in integrative approach to assess the key unit competence with cross reference to the student book.

Answers

1. a) The amount 50000 represents the fixed cost;

   b) The number 8 represents the the marginal cost (cost of a unit of product);

   c) The graph for $C(q) = 50000 + 7q$ is the following:



   d) The real domain of C that corresponds to q which is positive is $[0;\infty[$. The range is $[50000;\infty[$

   e) $C(q)$ is not an odd function because $C(-q) \neq -C(q)$.

2) a) $x = 13 \Rightarrow s = \{13\}$; b)$x \geq 4 \Rightarrow s = [4, +\infty[$; c)$s =]-3, 2[$;

   d) $s = \left]-\frac{7}{3}, 2\right[$; e) $s = \{5 - 2\sqrt{6}, 5 + 2\sqrt{6}\}$; f) $s = \left[\frac{1}{3}, \frac{1}{2}\right]$;

   g) $s = \{1\}$

3) Solving the equation $3\,000 + 20x = 7\,000$, we find that 200 minutes are on the bill.

4) a) TC=6000+25 x

b) TR= 50 x

c) P= 50 x -(6000+25 x) =25 x -6000

P =25×1000-6000=19,000

d) P =10 000 = 25 x -6000,

x = 16 000/25

x = 640 units

2.9. Additional activities

2.9.1. Remedial activities

Determine whether solutions exist for each of the following quadratic equations.

Where they do find the solution(s).

i) $x^2 - 2x = 0$

a=1, b=-2, c=0

$b^2 - 4a = (-2)^2 - 4(1)(0) = 4 > 0$ two solutions exist

$$x = \frac{-b \pm \sqrt{b^2 - 4a}}{2a} = \frac{2 \pm \sqrt{4}}{2(1)} = \frac{2 \pm 2}{2}$$

$$x = \frac{2+2}{2} = 2$$

$$x = \frac{2-2}{2} = 0$$

$S = \{0,1\}$

ii) $(3x-6)(x+1) = 0$

Multiply out the quadratic

$3x^2 - 3x - 6 = 0$

Divide across by 3

$x^2 - x - 2 = 0$

a=1, b=-1, c=-2

$b^2 - 4a = (-1)^2 - 4(1)(-2) = 9 > 0$

$$x=\frac{-b\pm\sqrt{b^2-4a}}{2a}=\frac{1\pm\sqrt{9}}{2(1)}=\frac{1\pm 3}{2}$$

$$x=\frac{1+3}{2}=2$$

$$x=\frac{1-3}{2}=-1$$

$$S=\{-1,2\}$$

2.9.2. Consolidation activities

Determine whether solutions exist for each of the following quadratic equations. Where they do find the solution(s).

i) $9x^2-24x+16=0$

a=9, b=-24, c=16

$b^2-4ac=(-24)^2-4(9)(16)=576-576=0$ one solution

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{24\pm\sqrt{0}}{2(9)}=\frac{24}{18}=1.33$$

ii) $3x^2+2x+3=0$

a=3, b=2, c=3

$b^2-4a=(2)^2-4(3)(3)=4-36=-32<0$ no solution, $S=\{\ \}$

iii) A firms demand function for a good is given by P = 107-2Q and their total cost function is given by TC = 200+3Q . find total revenue profit in terms of Q and values of Q does the firm break even

Solutions:

Obtain an expression for total revenue profit in terms of Q

Total Revenue = P.Q

TR = (107-2Q)*Q = 107Q-2Q^2

Profit = TR-TC

Profit = 107Q-2Q^2-200-3Q = -2Q^2+104Q-200

For what values of Q does the firm break even

Firm breaks even where Profit = 0

$-2Q^2+104Q-200 = 0$

a = -2, b=104, c=-200

$$Q=\frac{-104\pm\sqrt{(104)^2-4(-2)(-200)}}{2(-2)}=\frac{-104\pm\sqrt{10816-1600}}{-4}=\frac{-104\pm 96}{-4}$$

$$Q=2, Q=50$$

values of Q the firm breakeven are Q=2 and Q=50

2.9.3. Extended activities

Determine whether solutions exist for each of the following quadratic equations.

Where they do find the solution(s).

i) $-2x^2+x+10=0$

a=-2, b=1, c=10

$b^2-4ac=(1)^2-4(-2)(10)=81>0$ two solutions

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-1\pm\sqrt{81}}{2(-2)}=\frac{-1\pm 9}{-4}$$

$$x=\frac{-1+9}{-4}=-2 \qquad x=\frac{-1-9}{-4}=2.5$$

ii) The demand function for a goods given as $\mathbf{Q = 130-10P}$. Fixed costs associated with producing that good are **€60** and each unit produced costs an extra **€4**.

1. Find total revenue and total costs in terms of Q
2. For what values of Q does the firm break even
3. Sketch the graph of the profit function

1. Obtain an expression for total revenue and total costs in terms of Q

TR = P.Q

Q = 130-10P

10P = 130-Q

P = 13-Q/10

TR = (13-Q/10)*Q = 13Q-0.1Q2

TC = FC+VC

TC = 60+4Q

2. For what values of Q does the firm break even

Firm breaks even where TR = TC

$13Q-0.1Q^2=60+4Q$

$-0.1Q^2+9Q-60=0$

a=-0.1, b=9, c=-60

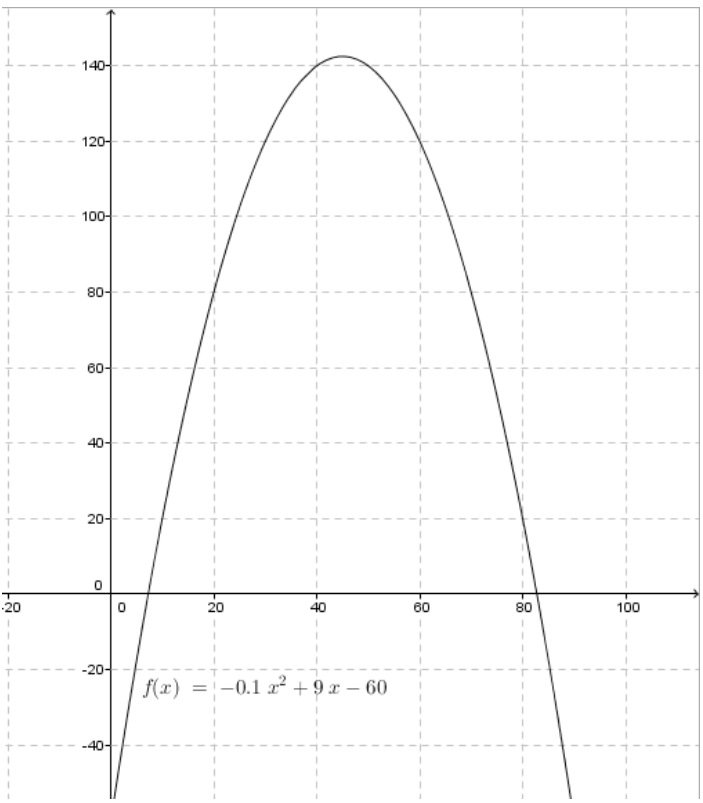
$$Q=\frac{-9\pm\sqrt{(9)^2-4(-0.1)(-60)}}{2(-0.1)}=\frac{-9\pm\sqrt{81-24}}{-0.2}=\frac{-9\pm 7.55}{-0.2}$$

$$Q=7.25, Q=82.75$$

3. Obtain an expression for profit in terms of Q and sketch its graph

Profit = TR-TC

Profit = $13Q-0.1Q^2-60-4Q=-0.1Q^2+9Q-60$



UNIT 3 EXPONENTIAL /LOGARITHMIC FUNCTIONS AND EQUATIONS

3.1. Key unit competence

Solve production, financial and economical related problems using logarithmic / exponential functions and equations

3.2. Prerequisites

Students will easily learn this unit, if they have a good background on numerical functions, equations and inequalities.

3.3. Cross-cutting issues to be addressed

- **Inclusive education:** Promote education for all during teaching process;
- **Peace and value Education**: Respect others' view and thoughts during class discussions
- **Gender**: Provide equal opportunity for both boys and girls during the lesson;
- **Environment and Sustainability:** During the lesson about population growth and growth rates, guide students to discuss the effect of the high rate of population growth;
- **Financial education**: Guide students to discuss how to manage loans taken from the bank looking at compound interest and depreciation;

3.4. Guidance on introductory activity 3

- Invite students to work in group discussion and give them instructions on how they can do the introductory activity found in unit 3 of student's book;
- Walk around all groups to provide pieces of advice where necessary.
- After a given time, invite students to present their findings and harmonize them.

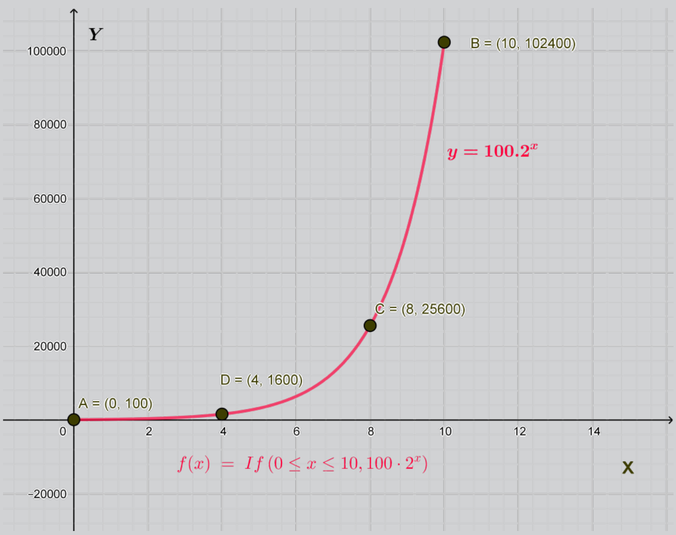
– Basing on students' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answer for introductory activity 3:

a) Student-teachers complete the table showing the money of the businessman from the 1st day up to 10th day.

Days	Amount	USD
1st days	200USD	200
2nd day	$200x2 = 100x2x2 = 100x2^2$	400
3rd days	$100x2x2x2 = 100x2^3$	800
4th days	$100x2x2x2x2 = 100x2^4$	1600
....		
10th day	$100x2x2x2x2... = 100x2^{10}$	102,400
Nth day	$100x2x2x2x2... = 100x2^n$	$100x2^n$

b) The graph plotted in a rectangular coordinate.



c) $f(n) = 100x2^n$ USD

- During the presentation let student-teachers discover the concept of exponential function $F(t)$ starting with the property of a function with powers. $F(t) = 100 \times 2^t$

- Student-teachers establish the function $Y(F)$ inverse of $F(t)$

$$Y(F) = F^{-1}(t) = \ln\left(\frac{t}{100}\right) = -\ln(100) + \ln t$$

$$Y(t) = -4.6 + \ln(t)$$

d) The economist wants to possess the money F under the same conditions, discuss how he/she can know the number of days necessary to get such money from the beginning of the business.

The economist wants to possess the money F, using the inverse function $Y(F) = -4.6 + \ln(F)$, she/he will use the equation $t = -4.6 + \ln(F)$ to calculate the number t of days required.

Conclude that $F(t)$ and $Y(t)$ are respectively exponential function and logarithmic functions that are needed to be well studied so that they may be used without problems. This unit deals with the behavior and properties of such essential functions and their application in real life situation.

3.5. List of lessons

Headings	#	Lesson title/sub-headings	Learning objectives	Number of periods
3.1. Exponential and logarithmic functions	0	Introductory activity	To arouse the curiosity of student-teacher on the content of unit 3.	1
	1	Exponential functions	Find out the domain and the range of exponential functions	2
	2	Decimal logarithmic functions	Find out the domain and the range of decimal logarithmic functions	3
	3	Natural logarithmic functions	Find out the domain and the range of Natural logarithmic functions	3

3.2. Exponential and logarithmic equations	1	Exponential equations	Use the properties of exponents to solve exponential equations.	3
	2	Decimal logarithmic equations	Solve decimal logarithmic equations	3
	3	Natural logarithmic equations	Solve Natural logarithmic equations	2
3.3. Applications of exponential and logarithmic functions	1	Growth rates	Use logarithmic and exponential properties to solve mathematical problems that involve production, finance and economics.	3
	2	Compound interest	Solve mathematical problems involving exponential and logarithmic functions	3
3.4. End unit assessment				1

Lesson 1: Exponential functions

a) Learning objectives

Find out the domain and the range of exponential functions

b) Teaching resources

Student's books and other reference textbooks, ruler, T-square, scientific calculator; if possible, mathematical software such as Geogebra and internet.

c) Prerequisites/Revision/Introduction

Students will easily learn this unit, if they have a good background on numerical functions and previous lesson of this unit

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 3.1.1 found in the Student's book;
- Move around in the class for facilitating students where necessary
- Facilitate students to establish the domain of the functions in the form of $g(x)=e^x$, and $h(x)=3^x$ then find out their ranges.
- During group discussions, move around to each group and prompt them to discuss the domain and range of the function $p(x)=a^x$ in case $a>0, a\neq 1$ and $a=1$.
- Lead students to harmonize the results by generalizing how to find the domain and the range of the function $f(x)=a^{u(x)}$ where $u(x)$ is a function of x.
- Guide students to work through examples in their books and work individually application activities 3.1.1. to assess the competences.

Answer for activity 3.1.1.

Consider the function $h(x)=3^x$ and complete the following table

X	-10	-1	0	1	10
$h(x)=3^x$	$\frac{1}{3^{10}}$	$\frac{1}{3}$	1	3	3^{10}

a) $\forall x\in\mathbb{R}, h(x)\in\mathbb{R}^+$, the domain of $h(x)$ is $\mathbb{R}=]-\infty,+\infty[$

b) All values $h(x)$ are positive, therefore, the range of $h(x)$ is $\mathbb{R}^+=]0,+\infty[$.

e) Answers for application activity 3.1.1.

1) $f(x)=5e^{2x}$,

$\forall x\in\mathbb{R}, f(x)x\in\mathbb{R}^+$, we realize that $domf=]-\infty,+\infty[$ and the range is the interval $]0,+\infty[$

2) $g\left(x\right)=3^{\left(\frac{x+1}{x-2}\right)}$

Condition for the existence of $\frac{x+1}{x-2}$ in $\mathbb{R}$: $x \neq 2$.

Therefore, $Dom\ g=\mathbb{R}\setminus\{2\}=]-\infty,2[\cup]2,+\infty[$ and range = $]3,+\infty[$

3) $3^{(x+4)}$, $\forall x\in\mathbb{R}, f(x)\in\mathbb{R}$, we realize that the $Domf=]-\infty,+\infty[$ and range $]0,+\infty[$

Lesson 2: Decimal logarithmic functions

a) Learning objectives

- Find out the domain and the range of decimal logarithmic functions
- State the restrictions on the base and the variable in a decimal logarithmic function

b) Teaching resources

Student's book and other reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will easily learn this unit, if they have a good background on numerical functions and previous lesson of this unit.

d) Learning activities:

- Invite students to work in groups and do the activity 3.1.2. found in student's book;
- Move around in all groups for facilitating students where necessary and give more clarification while discussing the given activity 3.1.2.
- Verify and identify groups with different working steps;
- Invite one person from each group who has various working steps to present their work and explain the steps.
- As teacher, harmonize the findings from presentation and guide students to plot the graph of $\log_{10}\left(x\right)$ for $x>0$

- Encourage students to share their thoughts on the graph they discovered step by step for $x>1, x=1, 0<x<1$ and values $x<0$
- Use different probing questions and guide students to explore the content and examples given in the student's book
- Lead students to discover the definition of decimal logarithmic function and determine the domain, and range of decimal logarithmic function;
- Facilitate students to deduce the domain and the range for $f(x)=\log_{10}(x)$ then generalize for the function of type $y=\log_a(u(x))$ with $u(x)\geq 0, a\neq 0, a>1$ and then from their answers, write a short summary.
- After this step, guide students to do the application activity 3.1.2 and evaluate whether lesson objectives were achieved.

Answer for activity 3.1.2

a) Complete the table of values for $\log_{10}(x)$

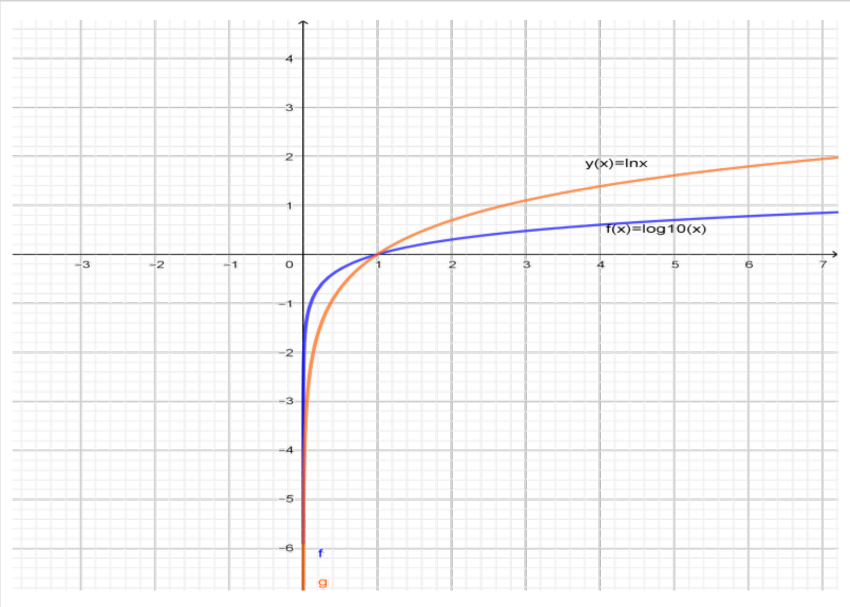
x	100	50	40	20	10	½	0.8	0.7	-5	-20	-30
$\log_{10}(x)$	2	1.69	1.6	1.30	1	-0.30	-0.09	-0.15	-	-	-

b) The value of $\log_{10}(x)$ for $x<0$ do not exist in the set of real numbers.

c) Discuss the values of $\log_{10}(x)$ for $0<x<1, x=1$, and $x>1$.

$\log_{10}(1)=0$, $\log_{10}(x)<0\, for\, 0<x<1$ and $\log_{10}(x)>0\, for\ x>1$

d) The graph of $\log_{10}(x)$ for $x>0$



e) For $f:\mathbb{R}\to\mathbb{R}: x\mapsto \log_a x$,

$dom\ f=\{x\in\mathbb{R}: x>0\}=\left]0,+\infty\right[=\mathbb{R}_0^+$ and range $f=\mathbb{R}=\left]-\infty,+\infty\right[$

e) Answers for application activity 3.1.2.

1. a) $y = \log_3(x-2)+4$ is defined for $x > 2$.
   $Domf =]2,+\infty[$

   To find the range we proceed as following:
   $y = \log_3(x-2)+4 \Leftrightarrow y-4 = \log_3(x-2)$ (for x in the domain)
   $\Leftrightarrow x-2 = 3^{y-4} \Leftrightarrow x = 3^{y-4}+2$

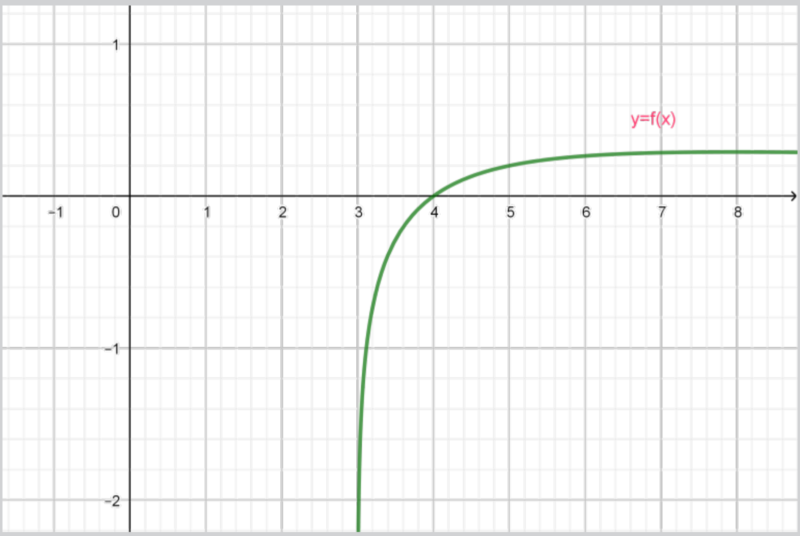
   Since $3^{y-4} > 0\ \forall y \in \mathbb{R}$, we have $x = 3^{y-4}+2 > 2$.

   Thus the range is $\mathbb{R}$

   b) $y = \log_5(8-2x)$ is defined only if $8-2x > 0 \Leftrightarrow -2x > -8 \Leftrightarrow x < 4$
   $Domf =]-\infty, 4[$

   For the Range:
   $y = \log_5(8-2x) \Leftrightarrow 8-2x = 5^y \Leftrightarrow -2x = 5^y - 8 \Leftrightarrow x = 4-5^y$
   $5^y > 0$ for all values of y implies $x = 4-5^y < 4$.
   Thus, the range is $\mathbb{R}$.

2. From the graph



The domain of the function f is $Domf =]3,+\infty[$. The range is $\mathbb{R} =]-\infty,+\infty[$

Lesson 3: Natural logarithmic functions

a) Learning objectives

- Find out the domain and the range of Natural logarithmic functions
- State the restrictions on the base and the variable in a Natural logarithmic function

b) Teaching resources

Student's book and other reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will easily learn this unit, if they have a good background on numerical functions and decimal logarithmic functions.

d) Learning activities:

- Invite students to work in groups and do the activity 3.1.3. found in student's book;
- Move around in all groups for facilitating students where necessary and give more clarification while discussing the given activity 3.1.3.
- Verify and identify groups with different working steps;
- Invite one person from each group who has various working steps to present their work and explain the steps.
- As teacher, harmonize the findings from presentation and guide students to plot the graph of $\log_{10}(x)$ for $x>0$ and $y(x)=\ln(x)$
- Encourage students to share their thoughts on the graph they discovered step by step.
- As teacher, harmonize the findings and maintain that the natural logarithmic function $y=\ln x$ is defined on positive real numbers, $]0,+\infty[$ and its range is all real numbers.
- Use different probing questions and guide students to explore the content and examples given in the student's book
- Lead students to discover the definition of natural logarithmic function and determine the domain, and range of natural logarithmic function;
- Facilitate students to deduce the domain and the range for $f(x)=\ln(x)$ then generalize for the function of type $y=\ln u(x)$

- After this step, guide students to do the application activity 3.1.3 and evaluate whether lesson objectives were achieved.

Answer for activity 3.1.3

1. a) The function $g(x)=\ln(6+x)$, is defined if and only if $6+x>0 \Leftrightarrow x>-6$ and gives that $x\in\,]-6,+\infty[$ which is the domain. The range is $\mathbb{R}$

   b) The domain of f consists of all x for which $(x-5)^2>0$. Thus, the domain of f is given by $\{x|x\neq 5\}$ or $]-\infty,5[\cup]5,+\infty[$ and Range is given by $]-\infty,+\infty[$ or $\mathbb{R}$

2. The given graph shown is the function defined as:

   $Domf=\,]-3,+\infty[$, the range of f is $]-\infty,+\infty[$ or $\mathbb{R}$

e) Answers for application activity 3.1.3.

a. The function $f(x)=\ln(4-x)$, is defined if and only if $4-x>0$ and gives that $]-\infty,4[$ is the domain. The range is $\mathbb{R}$

b. The domain of h consists of all x for which $x^2>0$. Thus, the domain of h is given by $\{x|x\neq 0\}$ or $]-\infty,0[\cup]0,+\infty[$ and Range is given by $]-\infty,+\infty[$ or $\mathbb{R}$

c. The domain of f consists of all x for which $(-4+x)^2>0$. Thus, the domain of f is given by $\{x|x\neq 4\}$ or $]-\infty,4[\cup]4,+\infty[$ and Range is given by $]-\infty,+\infty[$ or $\mathbb{R}$

Lesson 4: Exponential equations

a) Learning objective

Use the properties of exponents to solve exponential equations.

b) Teaching resources:

Scientific calculators to evaluate exponentials, student's book, graph papers. If possible, the use of Mathematical software such as Geogebra to plot graph and internet to facilitate research would be plausible.

c) Prerequisites/Revision/Introduction:

Students will perform well in this unit if they are enough skilled on: indices, radicals, solving equations learnt in ordinary level and exponential functions of the previous lesson

d) Learning activities:

This lesson deals with solving exponential equations.

- Form groups and ask students to work on activity 3.2.1.
- Ask randomly some groups to present their findings to the whole class;
- Lead students to give comments on previous presentation before the next one.
- Facilitate students to discuss the domain of validity/condition of existence.
- Facilitate students to apply exponential properties and use them to solve the exponential equations and then after their answers, write a short summary.
- Guide students to explore the content in student's book and work on the given examples
- Call students to do individually the application activities 3.2.1. to evaluate their competences.

Answers for activity 3.2.1.

For which value(s), each function $f(x)$ below can be defined. Explain.

1. $a) f(x)$ is defined for real numbers.

   $b) f(x) = e^{x^2-5x+6}$ is defined in $\mathbb{R}$.

2. $\ln 2^{1-x} = \ln 6$

$(1-x)\ln 2 = \ln 6$

$\ln 2 - x\ln 2 = \ln 6$

$x = \dfrac{\ln 2 - \ln 6}{\ln 2}$

e) Answers for application activity 3.2.1.

1) Solve each equation for x or t.

a)$5+e^{0.2t} = 10 \Leftrightarrow t = 5\ln 5$

b)$e^{2x} = 3e^{x} \Leftrightarrow (e^{x})^2 - 3e^{x} = 0$

$let\ e^{x} = t \Rightarrow (t)^2 - 3t = 0 \Leftrightarrow t = 0\ or\ t = 3$

$For\ t = 0, e^{x} = 0,$ No solution

$For\ t = 3, e^{x} = 3, then\ x = \ln 3$

c)$e^{2x} = e^{x} + 12 \Leftrightarrow (e^{x})^2 - e^{x} - 12 = 0$

By letting $e^{x} = y \Rightarrow (y)^2 - y - 12 = 0$

$then\ (y)^2 - y - 12 = 0 \Leftrightarrow y = 4\ or\ y = -3$

- $For\ y = 4 \Rightarrow x = \ln 4$
- $For\ y = -3 \Rightarrow no\ solution$

2) $)2e^{-x+1} - 5 = 9 \Rightarrow (-x+1) = \ln 7 \Rightarrow x = 1 - \ln$

$b)\dfrac{50}{1+12e^{-0.02x}} = 10.5 \Leftrightarrow 50 - 10.5 = 126e^{-0.02x}$

$e^{0.02x} = \dfrac{126}{39.5} \Rightarrow x = 50\ln\dfrac{126}{39.5}$

$c)\ e^{\ln x^2} - 9 = 0 \Leftrightarrow \ln x^2 = \ln 9$

$\therefore x = \pm 3$

$d)e^{x} - 12 = \dfrac{-5}{e^{-x}}$

3) $\dfrac{e^{2x}+1}{2} = 1e^{x} \Leftrightarrow e^{2x} - 2e^{x} + 1 = 0$

$let\ e^{x} = h \Rightarrow h^2 - 2h + 1 = 0 \Leftrightarrow h = 1$

$then\ x = 0$

4) Find the value of marked letter in each equation.

$a) 9^t + 3^t = 12 \text{ by letting } 3^t = b$

$b^2 + b - 12 = 0 \Rightarrow b = -4 \text{ or } b = 3$

$\bullet \text{For } b = -4 \text{; no solution.}$

$\bullet \text{For } b = 3; t = \ln 3$

b) $2^x + 2^{x-1} = \frac{3}{2}$ solution is given by $x = 0, S = \{0\}$

$c) \frac{2^x}{4} - \frac{3^x}{9} = 0 \Leftrightarrow 2^{x-2} = 3^{x-2} \Leftrightarrow (x-2)\ln 2 = (x-2)\ln 3$

$\Leftrightarrow x = \frac{2(\ln 2 - \ln 3)}{\ln 2 - \ln 3} = 2$

d) $\left(\frac{5}{2}\right)^x = 0.16$ solution is given by x=-2 and S={-2}

5) Solve

a. $2^{4x} - 6.2^{3x} + 6.2^x - 1 = 0$

Replace 2^x by t to form new equation $t^4 - 6t^3 + 6t - 1 = 0$

Solving the equation $t = -1 \, or \, t = 1$ or $t = 3 + 2\sqrt{2} \, or \, t = 3 - 2\sqrt{2}$

Since we have $2^x = t$ we apply natural logarithm to find out x,

$x = 0, x = \frac{\ln(3 + 2\sqrt{2})}{\ln 2}, x = \frac{\ln(3 - 2\sqrt{2})}{\ln 2}$

For (b), (c) and (d) use the same strategy in (a)

Lesson 5: Decimal logarithmic equations

a) Learning objective

Use the properties of logarithms to solve logarithmic equations.

b) Teaching resources:

Scientific calculators to evaluate logarithms, textbooks, graph papers. If possible, the use of Mathematical software such as Geogebra to plot graph and internet to facilitate research would be plausible.

c) Prerequisites/Revision/Introduction:

Students will perform well in this unit if they have enough skills on properties and operations common logarithms learnt in previous lessons, solving equations and inequalities in the set of real numbers, domain and range of logarithmic functions and decimal logarithmic function

d) Learning activities:

- Form groups and ask student-teachers to work on the activity 3.2.2.
- Ask randomly some groups to present their findings to the whole class;
- Lead students to give comments on previous presentation before the next one.
- Facilitate students to discuss the domain of validity/condition of existence.
- Facilitate students to apply logarithmic properties and use Changing base to another to solve the logarithmic equations and then after their answers, write a short summary.
- Guide students to explore content and work out on the given examples within student's book.
- Call students to do individually the application activities 3.2.2. to evaluate their competences.

Answers for activity 3.2.2.

1. Use the properties for logarithm to determine the value of x in the following expressions:

   a) $\log x = 2; x > 0$

   $\log x = \log 10^2$

   $x = 100$

   b) $\log(100x) = 2\log 10 + \log 4; \quad x > 0$

   $\log 100x = \log 400$

   $x = 4$

   c) $\log_2 x = -3$, $x > 0$ then, $x = \frac{1}{8}$

2. With the expression $\log_2 x = 5 - \log_2(x+4)$

   $x = 2^{5+\log_2(x+4)} \Leftrightarrow x = 2^5 \times 2^{\log_2(x+4)}$ the equation become

   $x^2 + 4x - 32 = 0$ solving the equation, the values of x are $x = 4$ and $x = -8$ this value is to be rejected as $-8 \notin]0, +\infty[$

e) Answers for application activity 3.2.2.

1. $i)\log(x+2)=2; \quad x+2>0; \quad x>-2$

   $\log(x+2)=\log 10^2$ then $x=98$; therefore $S=\{98\}$

   $b)\log x+\log(x^2+2x-1)-\log 2=0; \quad x>0, \quad x^2+2x-1>0$

   $S=\{1\}$

   $c)\log(35-x^3)=3\log(5-x)$

   $\log(35-x^3)=\log(5-x)^3$

   $35-x^3=125-75x+15x^2-x^3$

   $15x^2-75x+90=0$

   $x^2-5x+6=0$

   $x=3 \text{ or } x=2 \quad \therefore S=\{2,3\}$

   $d)\log(1-x)=-1; \quad 1-x>0 \Leftrightarrow x<1$

   $\log(1-x)=\log 10^{-1}$

   $x=\frac{9}{10} \qquad \therefore S=\left\{\frac{9}{10}\right\}$

   $e)\log(3x-2)+\log(3x-1)=\log(4x-3)^2; \quad x>\frac{2}{3}$

   $(3x-2)(3x-1)=(4x-3)^2$

Then solving the equation and respecting the condition, we have

$S=\left\{\frac{15+\sqrt{29}}{14}\right\}$

2. a. $\begin{cases} x+y=9 \\ \log x+\log y=\log 14 \end{cases}$

solution:

$$\begin{cases} x+y=9 \\ \log x+\log y=\log 14 \end{cases}$$

From equation (1), $x+y=9 \Rightarrow y=9-x$ replace in equation (2) $\log x+\log y=\log 14$ we get $\log x+\log(9-x)=\log 14$ applying the rules of logarithm we get $x(9-x)=14$ solve for x we obtain $x=2 \text{ or } x=7$ and $y=7 \text{ or } y=2$

b. $\begin{cases} x^2+y^2=221 \\ \log_5 x+\log_5 y=\log_5 110 \end{cases}$ from the system of equations

write the system as $\begin{cases} x^2+y^2=221 \\ xy=110 \end{cases}$ solve the system

$x=-10, x=10$ *and* $y=-11$ *or* $y=11$ reject $x=-10$ *and* $y=-11$ then accept $x=10$ *and* $y=11$, hence $S=\{10,11\}$

Lesson 6: Natural logarithmic equations

a) Learning objective

Use the properties of natural logarithms to solve natural logarithmic equations.

b) Teaching resources:

Scientific calculators to evaluate natural logarithms, student' s books, graph papers. If possible, the use of Mathematical software such as Geogebra and internet to facilitate research would be plausible.

c) Prerequisites/Revision/Introduction:

Students will perform well in this unit if they have enough skills on **properties and operations** natural logarithms learnt in previous lessons, solving equations and inequalities in the set of real numbers, domain and range of logarithmic functions and natural logarithmic function

d) Learning activities:

- Form groups and ask student-teachers to work on the activity 3.2.3.
- Ask randomly some groups to present their findings to the whole class;
- Lead students to give comments on previous presentation before the next one.
- Facilitate students to discuss the domain of validity/condition of existence.
- Facilitate students to apply natural logarithmic properties and use Changing base to another to solve the natural logarithmic equations and then after their answers, write a short summary.
- Guide students to explore content and work out on the given examples within student's book.
- Call students to do individually the application activities 3.2.3. to evaluate their competences.

Answers for activity 3.2.3.

1. The value(s) of x in which each function is defined:

   $a)\ln x = 0; \quad x > 0$

   $\ln x = \ln 1 \qquad\qquad Domf =]0,+\infty[$

   $x = 1$

   $b.h(x) = \ln(x+2)$, is defined on $x+2>0 \Leftrightarrow x>-2$

   Domain of validity is given by $x \in]-2,+\infty[$

   c. f(x) is defined on $]-\infty,2[\cup]3,+\infty[$
2. The value of x in the following expressions:

   $a.\ln x = 10 \Leftrightarrow x = e^{10} \qquad\qquad b.\ln x = 3 \Leftrightarrow x = e^3$

e) Answers for application activity 3.2.3.

$a)\ln x = 0; \quad x > 0$

$\ln x = \ln 1$

$x = 1$

b) $\ln\left(x^2-1\right) = \ln\left(4x-1\right) - 2\ln 2$

Domain of validity : $x \in]1,+\infty[$

$$\ln\left(x^2-1\right) = \ln\frac{(4x-1)}{4} \Leftrightarrow x^2-1 = \frac{(4x-1)}{4} \Leftrightarrow 4x^2-4 = 4x-1$$

$4x^2-4x-3=0$, solve this equation we get $x = \frac{3}{2}$ *and* $x = -\frac{1}{2}$

c) $2\ln 4\mathrm{x} = 7$, The value of $x = \frac{e^3\sqrt{e}}{4}$

d) $\ln x + \ln 4 = 0; \quad x > 0$

$\ln x = \ln 4^{-1}$

$x = \frac{1}{4}$

$e)\ln 2x = \ln 2.4 \qquad\qquad \begin{aligned} 2x &= 2.4 \\ x &= 1.2 \end{aligned}$

Lesson 7: Applications of exponential and logarithmic functions

a) Learning objective

Use logarithmic and exponential functions to solve mathematical problems that involve production, finance and economics.

b) Teaching resources:

Scientific calculators to evaluate decimal and natural logarithms, student' s books, graph papers. If possible, the use of Mathematical software such as Geogebra and internet to facilitate research would be plausible.

c) Prerequisites/Revision/Introduction:

Students will do well in this lesson if they have enough skills on exponential, decimal, and natural logarithmic functions/equations.

They will also do well in this lesson if they have sufficient skills in the following areas: solving equations learned in previous lessons, range and domain of logarithmic functions, and natural logarithmic functions

d) Learning activities:

- Form groups and ask students to work on the activity 3.3.
- Ask randomly some groups to present their findings to the whole class;
- Lead students to give comments on previous presentation before the next one.

Answers for activity 3.3.

As the interest on a loan or deposit calculated on both the original principal amount and the accumulated interest from previous periods refer to compound interest, if you need to check your final value after 5 years use the following formula $A(t)=P\left(1+\frac{r}{n}\right)^{nt}$, where A(t) is accumulated amount at time t, P is the principal, r is the interest rate n is the number of times the interest is compounded each year, and t is the number of years since the investment was made.

Logarithmic functions are useful to this situation as we can apply logarithm to the formula $A(t)=P\left(1+\frac{r}{n}\right)^{nt}$ in order to find out t as the number of years since the investment was made.

e) Answers for application activity 3.3.

Exponential and logarithmic functions are useul in calculation of Risk assessment and growth rate such as population growth rate, interest rate, etc…

Note: Answers may vary and teacher harmonize them.

3.6. Summary of unit 3

A. Exponential functions/Equations

- Definition: $f(x) = a^x$, where $a > 0, a \neq 1$, ;this definition is used to determine the domain and the range of an exponential function
- For $\text{f(x)} = a^x$ the domain is $]-\infty, +\infty[$

Exponential equation can be solved by using logarithms under the four steps:

- Isolate the exponential expression
- Take logarithms for both sides
- Rewrite exponential side as a linear expression
- Solve the obtained equation and check the answer

B. Logarithmic functions/Equations

- Definition: $\log_a x = y \Leftrightarrow a^y = x$, where $a > 0, a \neq 1, x > 0$;this definition is used to determine the domain and the range of decimal logarithm.
- Formula for changing the base, from base a to base e: $\log_a x = \dfrac{\ln x}{\ln a}$

Decimal logarithmic equation in $\mathbb{R}$ is the equation containing the unknown within the logarithmic expression. To solve it the steps below are proceeded:

- Set existance conditions for solution(s) of equation.
- Express logarithms in the same base
- Use logarithmic properties to obtain:

  $\log_a u(x) = \log_a v(x) \Leftrightarrow u(x) = v(x)$; where $u(x)$ and $v(x)$ are the functions in x.
- Make sure that the value(s) of unknown verifies the conditions above.

Note: The properties of logarithms can be used to solve decimal logarithmic equations.

C. Applications of logarithmic and exponential functions

- Interest compounded n times per year, r: rate of annual interest, P:

Principal

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

- Interest compounded continuously: $A = Pe^{rt}$
- Depreciation formula used to find out how assets lose value due to age or use., $D = P\left(1-\frac{r}{100}\right)^{n}$, where D is the final value of asset, P is the initial value of the asset, r is

While in finding out how loan payments are applied to specific types of loans, amortization formula

$A = P\frac{i(1+i)^n}{(1+i)^n - 1}$, where A is Periodic payment amount,

P is the principal, i is periodic interest rate and n is total number of payments

Note: Emphasize that students will calculate these growth rates, amortization, compound interest in unit 5: Financial mathematics.

3.9. Additional Information for the teacher

As teacher, inform students that:

- The concepts of logarithmic functions and exponential functions can be taught interchangeably. Exponential functions can be taught first: $f(x) = a^x$, where $a > 0$ and $a \neq 1$
- Logarithmic functions are the inverses of exponential functions. The inverse of the exponential function $y = a^x$ is $x = a^y$. The logarithmic function $y = \log_a x$ is defined to be equivalent to the exponential equation $x = a^y$.
- $y = \log_a x$ is under the following conditions: $x = a^y$, $a > 0$, and $a \neq 1$. It is called the logarithmic function with base a
- The domain of a logarithmic function is real numbers greater than zero, and the range is real numbers. The graph of $y = \log_a x$ is symmetrical to the graph of $y = a^x$ with respect to the line $y = x$. This relationship is true for any function and its inverse.

3.8. End unit assessment

Answers

1. a. $10+e^{0.1t}=14$

$e^{0.1t}=4$ **taking natural logarithm of each side**

$0.1t=\ln 4 \Leftrightarrow t=10\ln 4$

b. $3+2\ln x=7$,

$2\ln x=4 \Leftrightarrow e^{\ln x}=e^{2}$

$x=e^{2}$

c. $(6.5)^{x}=44$

$x\ln 6.5=\ln 44$

$x=\dfrac{\ln 44}{\ln 6.5} \Leftrightarrow x\approx 2.02$

2. Solve the following system of equations

a. $\begin{cases} x-y=8 \\ \log_2 x-\log_2 y=1 \end{cases}$ From equation 1, $x=8+y$ replace in equation 2 we obtain $\log_2(8+y)-\log_2 y=1$

The answer should be $S=\left\{\dfrac{e^{2}}{\sqrt{2}},\dfrac{e}{\sqrt{2}}\right\}$ $\log_2\left(\dfrac{8+y}{2}\right)=\log_2 y \Leftrightarrow \dfrac{8+y}{2}=y$

Therefore, $x=16$, $y=8$

$S=\{16,8\}$

b) $\begin{cases} \ln x-\ln y=1 \\ \ln 2x+\ln y=3 \end{cases}$

3. The domain and range of the following function

a) $f(x)=\log_2(3x-2)$

$f(x)=\log_2(3x-2)$ is defined if and only if $(3x-2)>0 \Leftrightarrow 3x>2 \Leftrightarrow x>\dfrac{2}{3}$

Thus, $Domf=\left]\dfrac{2}{3},+\infty\right[$

From the function, $0<3x-2<+\infty$

Then, $-\infty<\log_2(3x-2)<+\infty$

Therefore, the range is $\mathbb{R} =]-\infty, +\infty[$

b) $f(x) = \ln(x^2 - 1)$

$f(x) = \ln\left(x^2 - 1\right)$ is defined if and only if $x^2 - 1 > 0 \Leftrightarrow (x-1)(x+1) > 0$

Therefore, $Domf =]-\infty, -1[\cup]1, +\infty[$

From the function, $0 < x^2 - 1 < +\infty$ then $-\infty < \ln\left(x^2 - 1\right) < +\infty$

Therefore, the range is $\mathbb{R} =]-\infty, +\infty[$

4. Solve the following equations

a. $\log(2x^2 + 3) = \log(x^2 + 5x - 3)$ $\quad 2x^2 + 3 = x^2 + 5x - 3$

$$x^2 - 5x + 6 = 0 \Leftrightarrow x = \frac{5 \pm \sqrt{25 - 24}}{2} = \begin{cases} x = 3 \\ x = 2 \end{cases}$$

b. $ln\, x\ (x - 3) = ln\ 10 \Leftrightarrow x^2 - 3x - 10 = 0$

$x^2 - 3x - 10 = 0 \Leftrightarrow (x+2)(x-5) = 0$

Solving the equation we obtain $x = -2\, or\, x = 5$, reject $x = -2$ and accept $x = 5$

5. a) To calculate the final value that will be paid, the following formula can be applied

A(t) = Pe^{rt}, where P is the principal, r is the interest rate, and t is the number of years since the investment was made. The function A(t) model the value of an investment made with continuous compounding.

b) This reduction refers to the depreciation, over time, these assets lose value. This loss of value is called depreciation. The depreciation formula can be applied after 15 years of reduction , $D = P\left(1 - \frac{r}{100}\right)^n$, where D is the final value of asset, P is the initial value of the asset, r is is the interest rate.

c. To estimate the amortization period and monthly payment amount the following formula can be applied for calculating amortization, $A = P\frac{i(1+i)^n}{(1+i)^n - 1}$, where A is Periodic payment amount, P is the principal, i is periodic interest rate and n is total number of payments.

3.9. Additional activities

3.9.1. Remedial activities

1. How can an exponential equation be solved?

2. When can the one-to-one property of logarithms be used to solve an equation? When can it not be used?

Solutions:

Solution 1: Determine first if the equation can be rewritten so that each side uses the same base. If so, the exponents can be set equal to each other. If the equation cannot be rewritten so that each side uses the same base, then apply the logarithm to each side and use properties of logarithms to solve.

Solution 2: The one-to-one property can be used if both sides of the equation can be rewritten as a single logarithm with the same base. If so, the arguments can be set equal to each other, and the resulting equation can be solved algebraically. The one-to-one property cannot be used when each side of the equation cannot be rewritten as a single logarithm with the same base.

3. Solve the following equations

   a. $e^{5x}=17$ b. $7e^{3x-5}+7.9=47$

Solution:

a. The solution is obtained by applying logarithm for both sides then, $x=\frac{\ln 17}{5}\approx 0.567$

b. The solution is obtained by applying logarithm for both sides then, $x=\frac{1}{3}\left(5+\ln\left(39.1/7\right)\right)\approx 2.240$

4. Find out the domain and range of the following functions

   a. $f(x)=2e^{3x+1}$

Solutions:

$$domf=\left[-\frac{1}{3},+\infty\right[$$

From the function, $-\infty<3x+1<+\infty$

Then, $0<e^{3x+1}<+\infty$

$$0 < 2e^{3x+1} < +\infty$$

Therefore, the range is $]0, +\infty[$

b. $f(t) = 4^{\sqrt{3t+1}}$

Solutions:

$f(t) = 4^{\sqrt{3t+1}}$ is defined if and only if $3t + 1 \geq 0 \Leftrightarrow t \geq \frac{-1}{3}$

$$domf = \left[-\frac{1}{3}, +\infty\right[$$

From the function, $0 \leq 3t + 1 < +\infty$

Then, $0 \leq \sqrt{3t+1} < +\infty$, $1 \leq 4^{\sqrt{3t+1}} < +\infty$

Therefore, the range is $[1, +\infty[$

c. Find the domain and the range of the function;

i) $f(x) = \log(x-1)$ ii) $f(x) = 4^{\sqrt{6x}}$

Solution

i) Domain: $]1, +\infty[$

Range: $\mathbb{R} =]-\infty, +\infty[$

ii) Domain: $[0, +\infty[$

Range: $\mathbb{R} = [1, +\infty[$

3.9.2. Consolidation activities

1. Suppose the function $f(x) = 2x - \ln x$, State the domain and range

Solution

Function $f(x) = 2x - \ln x$

The Domain $Domf =]0, +\infty[$ and the range is $\mathbb{R} =]-\infty, +\infty[$

2. $\log_2 x + \log_6 x = 3$

$$\Leftrightarrow \log_2 x + \frac{1}{\log_2 6}\log_2 x = 3$$

$$\Leftrightarrow \log_2 x + \frac{1}{\log_2 6}\log_2 x = 3$$

$$(\log_2 6)(\log_2 x) + \log_2 x = 3\log_2 6 \Leftrightarrow (1+\log_2 6)\log_2 x = 3\log_2 6$$

$$\log_2 x = \frac{3\log_2 6}{1+\log_2 6} \Leftrightarrow x = 2^{\frac{3\log_2 6}{1+\log_2 6}}$$

3. Solve the following equation

a. $\begin{cases} 2\ln x + 3\ln y = -2 \\ 3\ln x + 5\ln y = -4 \end{cases}$ $by\ u\sin g\ e\lim ination\ method;\ S = \left\{\left(e^2, \frac{1}{e^2}\right)\right\}$

b. $\begin{cases} \ln(xy) = 7 \\ \ln\frac{x}{y} = 1 \end{cases}$ $by\ u\sin g\ e\lim ination\ method;\ S = \left\{\left(e^4, e^3\right), \left(-e^4, -e^3\right)\right\}$

3.9.3. Extended activities

1. Solve the following equations

a. $4^{2x-1} = 8^{x+3} \Rightarrow \left(2^2\right)^{(2x-1)} = \left(2^3\right)^{(x+3)}$

Solution:

$2^{2(2x-1)} = 2^{3(x+3)} \Rightarrow 2(2x-1) = 3(x+3)$

If $a^u = a^v, then\ u = v$

$4x - 1 = 3x + 9 \Rightarrow x = 11$; The solution set is $S = \{11\}$.

b. $\dfrac{e^x + e^{-x}}{2} = 1$

Solution:

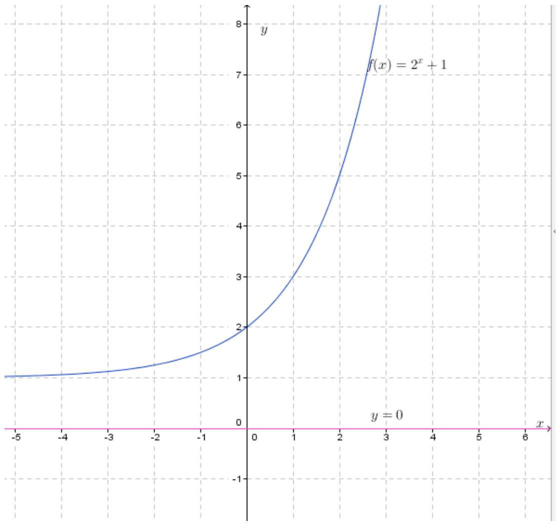
$$e^x\left(\frac{e^x + e^{-x}}{2}\right) = 1 \times e^x \Rightarrow \frac{e^{2x}+1}{2} = e^x$$

$\Leftrightarrow x = 0$

c. $2\log_2 x + \dfrac{1}{\log_2 x} = 3$

$x = 2$ or $x = \sqrt{2}$

2. The given graph



Find out the domain of definition and range

Solution:

Dom f = $domf =]-\infty,+\infty[$ or $domf = \mathbb{R}$ and range is $]1,+\infty[$

3. In $\mathbb{R}^2$, Solve the following system of equations

a) $\begin{cases} 5^{3x} = 25^{2y-2} \\ 9^{y} = 3^{x+1} \end{cases}$ Solution set S= $\left\{-2,-\frac{1}{2}\right\}$

b) $\begin{cases} 3^{x+1} = 243 \\ 2^{y} = 64 \end{cases} \Leftrightarrow \begin{cases} 3^{x+1} = 243 \\ 2^{y} = 64 \end{cases} \Leftrightarrow \begin{cases} 3^{x+1} = 3^{5} \\ 2^{y} = 2^{6} \end{cases} \Leftrightarrow \begin{cases} x+1 = 5 \\ y = 6 \end{cases} \Leftrightarrow S = \{(4,6)\}$

4. Suppose that you deposit P dollars in account whose annual interest r, compounded continuously. Help the bank manager to know how long will it take for the deposited amount to double.

Solution:

After t years $A = Pe^{rt}$. So, the balance will have double when $Pe^{rt} = 2P$, to find the doubling time solve the equation for t

$Pe^{rt} = 2P \Leftrightarrow e^{rt} = 2$

$t = \frac{1}{r}\ln 2$

UNIT 4

LIMIT OF NUMERICAL, LOGARITHMIC, EXPONENTIAL FUNCTIONS AND APPLICATIONS

4.1. Key unit competence

Apply limits in solving production, financial and economical related problems.

4.2. Prerequisite

Students will perform well in this unit if they are familiar with Numerical functions, Equations and Inequalities, , Logarithmic and Exponential Functions and equations.

4.3. Cross-cutting issues to be addressed

- Inclusive education (promoting education for all in teaching and learning process)
- Peace and value Education (respect the views and thoughts of others during class discussions)
- Gender: Provide equal opportunity for boys and girls to participate in class

4.4. Guidance on introductory activity 4

- Invite students to work in group, discuss and find out the answers for the introductory activity 4 from the student's book
- Facilitate students' discussions and ask them to avoid noise or other unnecessary conversations.
- During discussions, let students think of different ways to solve the given problem
- Walk around in all groups to provide assistance if necessary ;
- Invite group representatives to present their findings and encourage boys and girls to actively participate in presentations;
- Basing on students' experience, prior knowledge and abilities shown in answering the questions for this introductory activity 4, use different probing questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Example of probing questions

1. When the quantity, Q, approaches to 0, the price, P, gets "closer and closer "to what value?
2. When the quantity, Q, approaches to 20, the price, P, gets "closer and closer "to what value?
3. Refer to the graph. What happen on the price P if the quantity Q is getting smaller?

Answers for introductory activity 4

1) As $P = 200 - 0.24Q$, then

Q	1	0.5	0.1	0.01	0.001	0.0001	…	0
P	200	200	200	200	200	200		200

When Q approaches 0, the price gets closer and closer to 200.

2) The values of P when Q approaches 20 are given in the table below:

Q	19.5	19.9	19.9999	20	20.1	20.2	20.5	21
P	200	200	195.20024	195.2	195.176	195.152	195.08	194.96

When Q approaches 20, the price gets "closer and closer" to 195.2. This can be written as $\lim_{Q \to 20} P = \lim_{Q \to 20} (200 - 0.24Q) = 195.2$

Note: we rounded up.

4.5. List of lessons

Headings	#	Lesson title/ sub-headings	Learning objectives	Number of periods
4.1 Limits of functions	0	Introductory activity	Arouse the students' curiosity about limits of functions and their real life applications.	1
	1	Definition and neighborhood of real numbers	Explain the concept of limits for real-valued functions.	1
	2	One-sided limits, existence of limit	Apply informal methods to explore the concepts of one sided and impossible limits.	1

	3	Limits of numerical functions: Properties and operations	Determine the limits of polynomial , rational and irrational functions	2
	4	Graphical interpretation of limit of a function	Guess / determine the limit of a function basing on its graphical representation	1
	5	Squeeze theorem	Apply squeeze theorem to solve problems that involve limits	1
	6	Limits of exponential functions: Properties and operations	Perform operations on limits of exponential functions	1
	7	Limits of logarithmic functions: Properties and operations	Perform operations on limits of logarithmic functions	1
	8	Limits involving infinity and indeterminate cases	Determine limits that involve infinity and indeterminate cases	1
4.2. Applications of limits	1	Continuity of a function at a point or an interval	Apply limits to find out continuity of functions.	1
	2	Asymptotes to a curve or a graph of a function	Apply limits to determine vertical, horizontal, and oblique asymptotes of functions.	1
	3	Continuity and asymptotes of logarithmic functions	Apply limits to determine asymptotes of logarithmic functions.	1

	4	Continuity and asymptotes of exponential functions	Apply limits to determine asymptotes of exponential functions	1
4.3. End unit assessment				1

Lesson 1: Definition of Limit and Neighborhood of a Real Number

a) Learning objectives:

- Define the neighbourhood of a real number
- Find out the value of a function at a given point of the variable.
- Define the concept of limit for a real-valued function.

b) Teaching resources:

Student's book and other reference textbooks, Mathematical set, calculator, Manila paper, graph paper, ruler, markers, pens, pencils, different graphs on charts etc.

c) Prerequisites/Revision/Introduction:

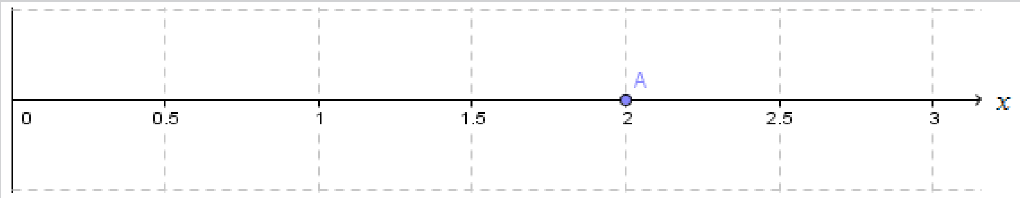
Students will perform well in this lesson if they are skilled enough in the content of numerical functions : plotting and interpretation of graphs, determination of the numerical value of a function, etc

d) Learning activities

- Invite Students to work in group and do the activity 4.1.1 inform the S4 Mathematics student's book;
- Move around for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Invite groups with different working steps to present their work;
- As a teacher, harmonize the group findings and guide them to explore the content and examples given in the student's book where they will be able to differentiate the neighbourhood of a real number from the value of a function at a given point and finally help them to mathematically define a limit of a function.
- Ask students to do the application activity 4.1.1 and evaluate whether lesson objectives were achieved to assess their competences

Answer for activity 4.1.1

a)



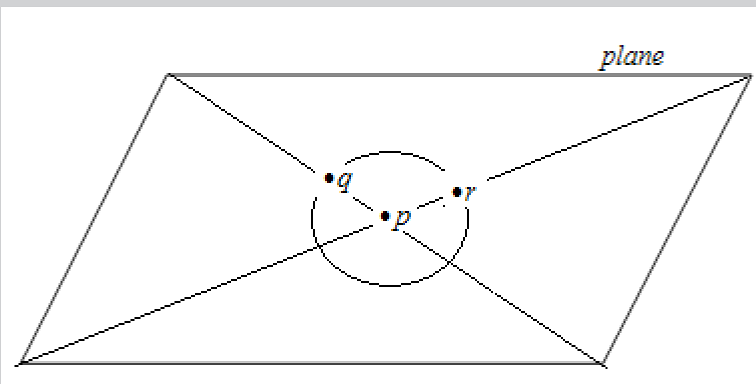
b) Open intervals for which $x=2$ is a center: $]0,4[$, $]1.9,2.1[$, $]1.99,2.01[$ and $]1,3[$.

c) The values given in the table show that when $x=2.00001$ or $x=1.99999$ approaches 2, $f(x)=3.99998$ or $f(x)=4.00001$ approaches the value 4 (i.e $f(x)=4$).

Therefore, as x approaches 2, the value of $f(x)$ approaches 4.

e) Answers of Application activity 4.1.1

1) Apart of Vatican in Europe, an example of African country that is surrounded by a single country is Lestho.
2) Three example of open intervals that are neighborhood of -5are: $]-6,-4[$, $]-7,-3[$, $]-10,0[$.
3) A circle is not a neighborhood of each of its points but, it is a neighborhood of it center but not for all its points.
4) The following plane is a neighborhood of points' p, q and r.



Lesson 2: One-sided limits, and existence of limits

a) Learning objective:

Evaluate one sided limits and their existence

b) Teaching resources

Student's book and other textbooks mathematical set, calculator, Manila paper, markers, pens, pencils, different graphs on charts, etc.

c) Prerequisites/Revision/Introduction

In this lesson, Students will perform better if they are enough skilled on graphs and numerical functions, exponential and logarithmic functions

d) Learning activities

- Invite Students to work in groups and do the activity 4.1.2 from S4 Mathematics student's book;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges students may face during their work;
- Invite groups to present their findings. At this stage, the technique of gallery work can help. Students may post their works on the wall and then they can make a tour to explore and get explanations from the group members. Common points from different groups are considered and new points are highlighted
- As a teacher, harmonize the findings from presentation and guide them to guess the true definition of limit of a function at a given point basing on the value of the left hand limit and the right hand limit.
- Use different probing questions and guide them to explore the content and examples given in the student's book lead them to be able to determine the limit of a function at a point by applying related properties.
- Assign students to do the application activity 4.1.2 and evaluate whether the lesson objectives were achieved

Answers for activity 4.1.2

a. $f(x)=\begin{cases}0 \text{ for x} < 0\\ 1 \text{ for x} \geq 0\end{cases}$ $\qquad g(x)=\begin{cases}0; \text{ for } x \leq 1\\ x; \text{ for } x > 1\end{cases}$

b. If we stay to the left side, as x approaches 0, $f(x)$ gets closer to 0

c. If we stay to the right side, as x approaches 0, $f(x)$ gets closer to 1

d. If we stay to the left side, as x approaches 1, $g(x)$ gets closer to 0

e. If we stay to the right side, as x approaches 1, $g(x)$ gets closer to 1

e) Answers for application activity 4.1.2

Given the $\lim\limits_{x\to 3} f(x)$ for $f(x)=\begin{cases}x^2-5 & \text{if } x \leq 3\\ \sqrt{x+13} & \text{if } x > 3\end{cases}$,

- As x approaches 3 from the left, the function f is $f(x)=x^2-5$. So that $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3}\left(x^2-5\right)=4$
- As x approaches 3 from the right, the function f is $f(x)=\sqrt{x+13}$. So that

$$\begin{aligned}\lim_{x\to 3^+} f(x) &= \lim_{x\to 3}\sqrt{x+13}\\ &= \sqrt{\lim_{x\to 3}(x+13)}\\ &= \sqrt{16}\\ &= 4\end{aligned}$$

Since the limits from left and right sides are equals, then limit of the function f(x) exist and it is written as follows: $\lim\limits_{x\to 3} f(x)=4$.

Lesson 3: Limits of numerical functions, properties and operations

a) Learning objective:

Evaluate the limit of numerical functions by applying their properties of operations.

b) Teaching resources

Student's book and other textbooks, mathematical set, calculator, manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

In this lesson, students will perform better if they are enough skilled on the definition of limit, neighborhood of a real number, one sided limit and existence of limit learnt in previous lessons

d) Learning activities

- Invite students to work in groups and do the activity 4.1.3 inform the student's book;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges students may face during their work;
- Invite students to present their findings;
- As teacher, harmonize the students findings and guide students to guess the properties of operations on Limits of numerical functions
- Use different probing questions and guide students to explore the content and examples given in the student's book. The probing questions are used to guide students on how to determine the limit of a numerical function at a point by applying appropriate properties of operations.
- After this step, assign students to do the application activity 4.1.3 and finally, provide the evaluation/assessment activities to check whether the lesson objectives were achieved.

Answers for activity 4.1.3

Evaluate

1) Given $f(x)=\frac{x+1}{x+2}$ then $f(2)=\frac{3}{4}$

2) Given $f(x)=\frac{\sqrt{x+3}}{\sqrt[3]{3x-2}}$ then $f(1)=2$

3) Given $f(x)=4x^3-2x^2+3x-1$, then $f(3)=4(3)^3-2(3)^2+3(3)-1=98$

e) Answers for application activity 4.1.3

1. Given that $-x^2\le g(x)\le x^2$. Find $\lim\limits_{x\to 0} g(x)$

   $\Rightarrow$ The limit of $g(x)$, when x tends to zero is equal to zero

2. If $\lim\limits_{x\to 3} f(x)=6$ and $\lim\limits_{x\to 3} g(x)=-3$. Find

   i. $\lim\limits_{x\to 3}\left[f(x)+g(x)\right]\Rightarrow$ *The answer is* 3 .This means $6-3=3$

   ii. $\lim\limits_{x\to 3}\left[f(x)g(x)\right]^3$

   $\Rightarrow$ *The answer is given by* $(6\times -3)^3=(-18)^3=-5{,}832$

3. Explain why the following calculation are incorrect

   a. $\lim\limits_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right)=\lim\limits_{x\to 0^+}\frac{1}{x}-\lim\limits_{x\to 0^+}\frac{1}{x^2}=\infty-\infty=0$

   The answer is incorrect because $\infty-\infty$ is an indeterminate form and it is different from zero.

   d. Show that $\lim\limits_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right)=-\infty$

   The answer is incorrect too because the right answer is indeterminate form $\infty-\infty$.

Lesson 4: Graphical interpretation of limit of a function

a) Learning objective:

Guess / determine the limit of a function given its graphical representation.

b) Teaching resources:

Student's book and other reference textbooks, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they have knowledge of numerical functions : plotting and interpretation of graphs, determination of the numerical value of a function, etc.

d) Learning activities

- Invite Students to work in groups and do the activity 4.1.4 inform the student's book;
- Move around in the class to provide guidance to the groups where necessary especially address the eventual challenges students may face during their work;
- Invite groups with different working steps to present their work;
- As a teacher, harmonize the findings from students and guide them to predict the limit they observe on a graph of function.
- In pairs, ask students to read the content and examples given in the student's book under the activity 4.1.4 in order to help them to determine by observation the limits of functions using their graphical representations.
- After this step, guide students to do the application activity 4.1.4 and evaluate whether lesson objectives were achieved to assess their competences

Answers for activity 4.1.4

1) $\lim_{x\to 1^-} f(x)=1$
2) $\lim_{x\to 1^+} f(x)=2$
3) Basing on the results from the left and right sided limits are not equal, the limit of $\lim_{x\to 1} f(x)$ doesn't exist.

e) Answers for Application activity 4.1.4

1. The limit $\lim_{x \to 2} f(x)$ is 4 **.**
2. $\lim_{x \to -1} h(x)$ doesn't exist, $\lim_{x \to 1} h(x)$ doesn't exist,
   $\lim_{x \to -\infty} h(x) = 1$ $\lim_{x \to \infty} h(x) = 1$

Lesson 5: Squeeze theorem

a) Learning objective:

Apply squeeze theorem to determine limits of numerical; functions

b) Teaching resources

Student's book and other reference textbooks, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

Students will perform well in this lesson if they have good background in lesson the previous contents and units.

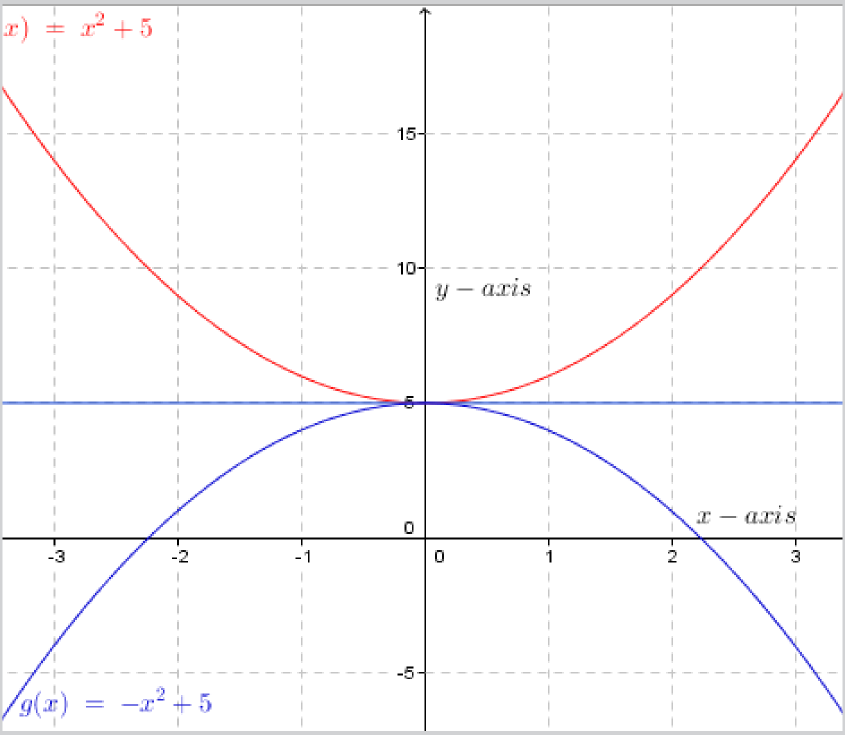
d) Learning activities

- Invite Students to work in groups and do the activity 4.1.5 the student's book;
- Move around in the class for facilitating students where necessary;
- Invite groups with different working steps to present their work.
- As a teacher, harmonize the findings from group presentations .
- Use different probing questions and guide students to read the content and examples given in the student's book
- After this step, ask students to do the application activity 4.1.5 and evaluate whether the lesson objective is achieved .

Answers for activity 4.1.5

From the figure we can see that if the limits of $f(x)$ and $g(x)$ are equal at $x = c$ then the function values must also be equal at $x = c$. However, because $h(x)$ is "**squeezed**" between $f(x)$ and $g(x)$ at this point then $h(x)$ must have the same value. Therefore, the limit of $h(x)$ at this point must also be the same. Similar statements hold for left and right limits.

e) Answers for the application Activity 4.1.5



The curve of $h(x)=5$ lies between other two curves and the three curves meet at the same point $(0,5)$. Therefore, $\lim_{x\to 0} f(x)=\lim_{x\to 0} g(x)=\lim_{x\to 0} h(x)=5$

Lesson 6: Limits of exponential functions

a) Learning objective:

Determine limits of exponential functions

b) Teaching resources

Student's book and other reference textbooks, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

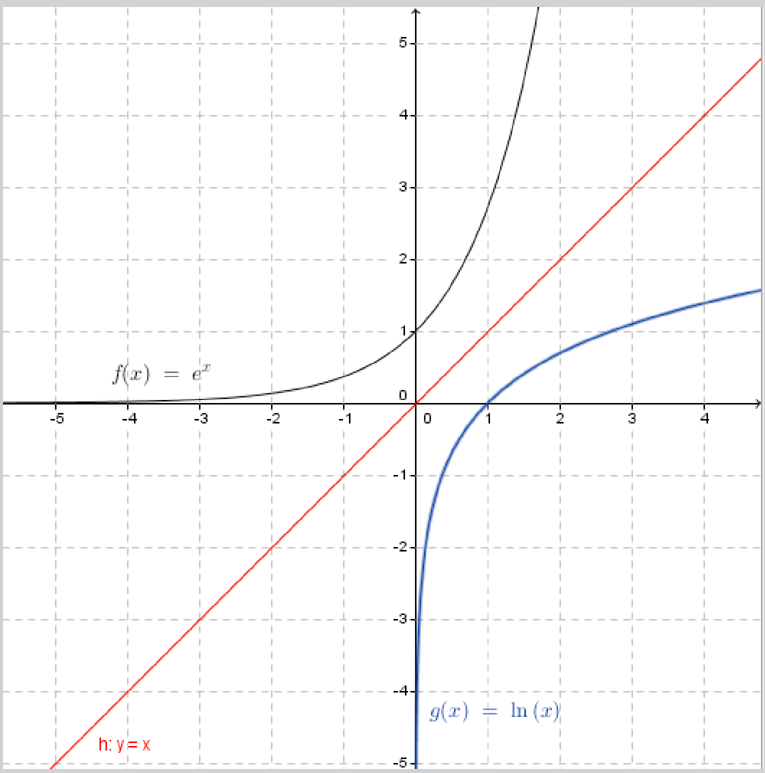
Students will perform well in this lesson if they have good background in lesson the previous contents and units.

d) Learning activities

- Invite Students to work in groups and do the activity 4.1.6 from the student's books;
- Move around in the class for facilitating students where necessary;
- Invite group representatives to present their work.
- As a teacher, harmonize the findings from group presentations and
- ask student to read the content and examples about limits of exponential functions and related to the activity 4.1.6 ;
- Ask students to do the application activity 4.1.6 sub- questions 1, 2 1nd 3 while su-questions 3 and 4 can be used as the assessment activities to to evaluate whether lesson objectives were achieved.

Answers for Activity 4.1.6

1) a) On the graph, the Functions $f(x)=\ln x.$ and its inverse $g(x)=f^{-1}(x)=e^{x}$ are symmetric on the diagonal line x=y. . The natural logarithm $f(x)=\ln x.$ is the inverse of exponential function $f^{-1}(x)=e^{x}$ as $x=e^{\ln x}$.

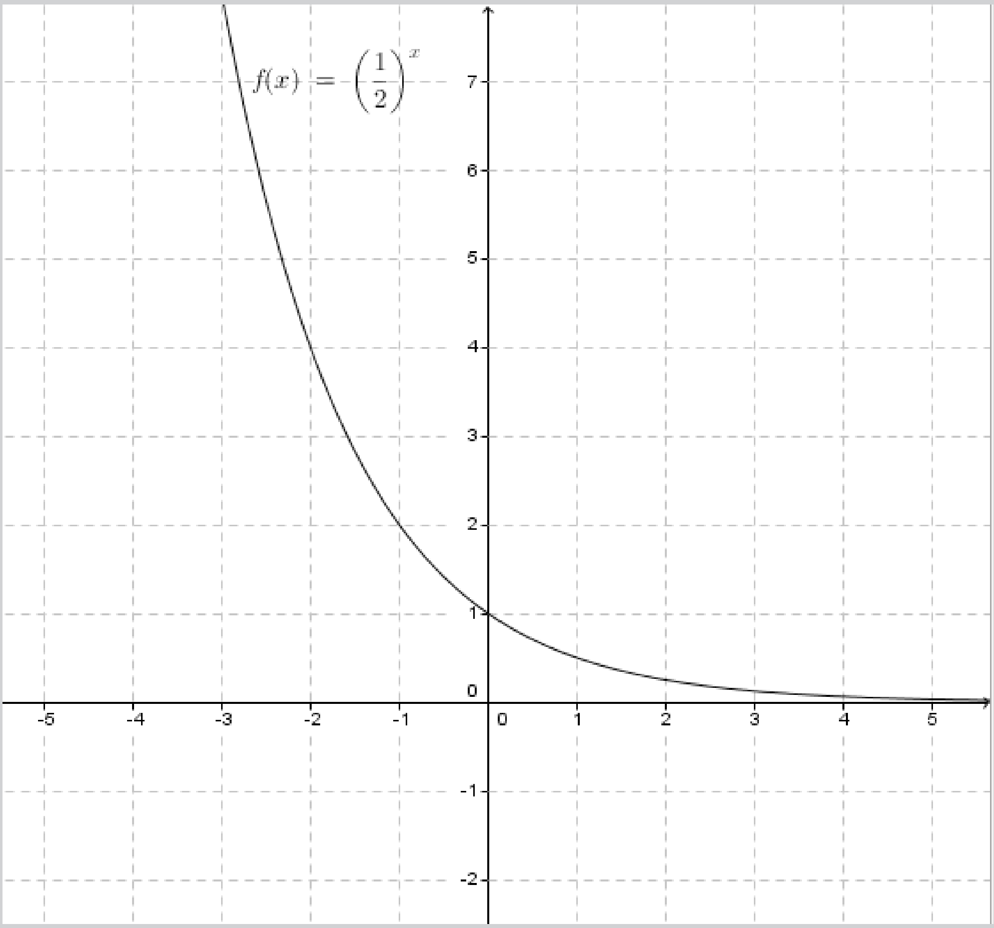


b)From the graph, it is clearly observed that the function $f(x)=e^x$ approaches the line $y=0$ when x takes bigger negative values and it goes to $+\infty$ when x takes bigger positive values. Hence,

$$\lim_{x\to-\infty} e^x = 0 \quad and \quad \lim_{x\to+\infty} e^x = +\infty$$

2) From the graph, it is clearly observed that the function $f(x)=\left(\frac{1}{2}\right)^x$ approaches the line $y=0$ when x takes bigger positive values and it goes to $+\infty$ when x takes bigger negative values . Hence,

$$\lim_{x\to-\infty}\left(\frac{1}{2}\right)^x = +\infty \quad and \quad \lim_{x\to+\infty}\left(\frac{1}{2}\right)^x = 0 \quad .$$



e) Answer for application Activity 4.1.6

For each function, evaluate limit at $+\infty$ and $-\infty$:

1. $f(x)=e^{8+2x-x^3}=e^{8+2(-\infty)-(-\infty)^3}=e^{8-\infty+\infty}=e^{8}\times e^{-\infty}\times e^{+\infty}=0\times(+\infty)$

2. $f(x)=e^{\frac{6x^2+x}{5+3x}}\Rightarrow f(-\infty)=\mathrm{e}^{-\infty}=0.$ and,

$f(x)=e^{\frac{6x^2+x}{5+3x}}\Rightarrow f(+\infty)=\mathrm{e}^{+\infty}=+\infty$

3. $f(x)=2e^{6x}-e^{-7x}-10e^{4x};\quad\Rightarrow f(-\infty)=2(e^{-6\infty})-e^{+7\infty}-10e^{-4\infty}=0-\infty+0=-\infty$

$\Rightarrow f(+\infty)=2(e^{6\infty})-e^{-7\infty}-10e^{4\infty}=\infty-0-\infty=\infty-\infty\ IF$

4. $f(x)=3e^{-x}-8e^{-5x}-e^{10x}\quad\Rightarrow f(-\infty)=+\infty-\infty+0=+\infty-\infty\ IF;\Rightarrow f(\infty)=0-0+\infty=+\infty$

5. $f(x)=\dfrac{e^{-3x}-2e^{8x}}{9e^{8x}-7e^{-3x}};\quad\Rightarrow f(-\infty)=\dfrac{\infty-0}{0-\infty}=\dfrac{\infty}{-\infty},\quad IF;\quad\Rightarrow f(+\infty)=\dfrac{0-\infty}{\infty-0}=\dfrac{-\infty}{\infty}\quad IF$

Notice: These indeterminate cases in exponential and logarithmic functions might be removed after studying the unit of derivatives in senior five.

Lesson 7: Limits of the logarithmic functions

a) Learning objective

Calculate limit of logarithmic functions with any base.

b) Teaching resources:

T-square, ruler, text book, if possible mathematical software such as Geogebra, Microsoft Excel, Mathlab.

c) Prerequisites/Revision/Introduction:

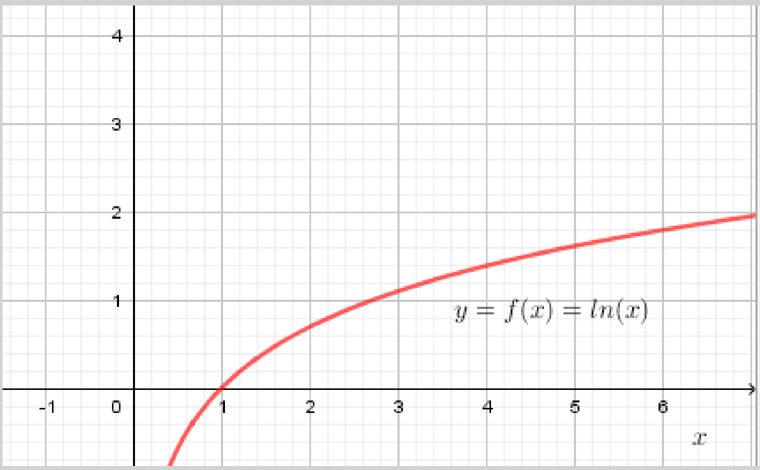
Students will perform well in this lesson if they have a good background on properties and operations of common logarithms and Natural logarithm or Naperian logarithm, solving equations and inequalities in the set of real numbers, domain and range of polynomial, rational , irrational, and logarithmic functions, limits of polynomial, rational and irrational functions.

d) Learning activities:

- Through group discussions invite student to do all questions of activity 4.1.7 and ask them to complete the given table and deduce limits of a given logarithmic function at the given point.
- Move around to ensure all students in groups participate actively.
- Call upon groups to present their findings.
- Harmonize their findings by leading students to calculate $\lim_{x\to+\infty} \log_a u(x)$.
- Let students read the content and examples related to the activity 4.1.7 and then work individually the application activities 4.1.7 to assess their competences.

Answers for activity 4.1.7

The graph below represents natural logarithmic function $f(x) = \ln x$



Using calculator and considering the form of the graph , we find that , for each value of x in the table, there is a value of $\ln x$ as shown in the table below,

x	0.5	0.01	0.001	0.0001	2	100	1001	10000
$\ln x$	-0.69	-4.61	-6.61	-9.21	0.69	4.61	6.91	9.21

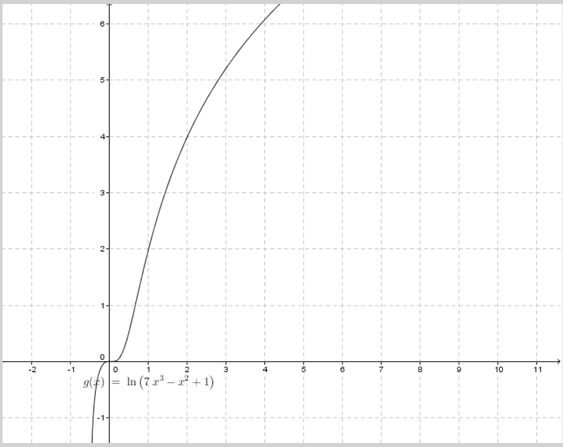
a) The values of $\ln x$ when x takes values closer to 0 from the right approach to minus infinity $(-\infty)$, hence $\lim_{x\to 0^+} \ln x = -\infty$.

b) The values of $\ln x$ when x take greater values go to plus infinity $(+\infty)$, hence $\lim_{x\to+\infty} \ln x = +\infty$

e) Answers of Application Activity 4.1.7

Evaluate the following limits

1) $\lim_{x\to+\infty} \ln(7x^3 - x^2 + 1) = \infty - \infty IF$ but with the graph , it is possible to guess the limit of the given function when the values of x are getting bigger.

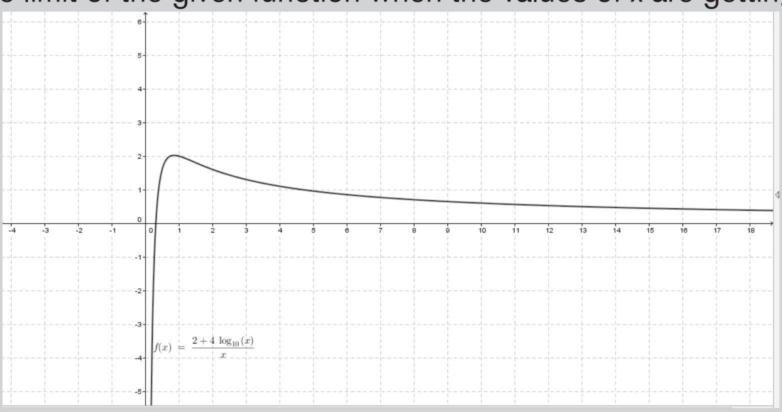


2)
$$\lim_{x\to1^+} \ln\frac{1}{x-1} = \ln\frac{1}{1^+ - 1} = \ln\frac{1}{0^+} = \ln(+\infty) = +\infty$$

3)
$$\lim_{x\to2^-} \log_5\left(x^2 - 5x + 6\right) = \log_5\left(\left(2^-\right)^2 - 5\left(2^-\right) + 6\right) = \log_5\left(4^- - 10^- + 6\right)$$
$$= \log_5\left(-6^- + 6\right) = \log_5\left(0^+\right) = -\infty$$

4)
$$\lim_{a\to4^+} \ln\frac{a}{\sqrt{a-4}} = ?$$
$$= \ln\frac{4^+}{\sqrt{4^+ - 4}} = \ln\frac{4^+}{\sqrt{0^+}} = \ln\frac{4^+}{0^+} = \ln 4^+\left(\frac{1}{0^+}\right) = \ln(+\infty) = +\infty$$

5) $\lim_{x\to+\infty} \frac{2 + 4\log x}{x} = \frac{\infty}{\infty} IF$, but with the graph , it is possible to guess the limit of the given function when the values of x are getting bigger.



Lesson 8: Limits involving infinity and indeterminate cases

a) Learning objective:

Perform operations on limits involving infinity.

b) Teaching resources

Students' book and other reference textbooks, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

Students will perform well in this lesson if they have good background information on properties and operations of common logarithms and Natural logarithm, domain and range of polynomial, rational , irrational, and logarithmic functions, limits of polynomial, rational and irrational functions.

d) Learning activities

- Invite students to work in groups and do the activity 4.1.8.1 inform their Mathematics books;
- Move around in the class for facilitating students where necessary and ask some challenging questions to lead them to successfully perform the given task;
- Verify and identify groups with different working steps;
- Invite groups to present their work.
- As a teacher, harmonize the findings from group presentations and guide on how to determine the limits involving infinity and indeterminate cases as it is given in the student's book.
- Use different probing questions, help students to read the examples and content related to the activity 4.1.8.1 in their book for better understanding on how to determine limits of functions involving infinity or indeterminate cases.
- After this step, ask students to do the application activity 4.1.8.1. sub-question a and b while sub-question c can be used to evaluate whether the lesson objectives were achieved.

Answers for activity 4.1.8.1

1. Given the function $f(x)=\dfrac{x+1}{x-1}$; we find that :

   a. $f(0.97)=-65$; b. $f(0.98)=-99$;

   c. $f(0.99)=-199$; d. $f(1.01)=201$; e. $f(1.02)=101$;

   f. $f(1.03)=67.6$

2. Evaluate the following operations:

   a) $-2+\infty=+\infty$ b) $2-\infty=-\infty$;

   c) $-\infty+\infty$ is an indeterminate form; d) $-\infty(\infty)=-\infty$;

   e) $3(-\infty)=-\infty$; f) $\dfrac{-\infty}{-2}=+\infty$; g) $\dfrac{\infty}{-\infty}$ is an indeterminate form.

e) Answers for application activity 4.1.8.1.

a. $\lim\limits_{x\to\infty}\dfrac{2x^2-3x+5}{x-2}=\lim\limits_{x\to\infty}\dfrac{2x^2}{x}=\lim\limits_{x\to\infty}2x=\lim\limits_{x\to\infty}2(\infty)=\infty$

b. $\lim\limits_{x\to\infty}\dfrac{2x^2+6}{3x^2-4x+2}=\lim\limits_{x\to\infty}\dfrac{2x^2}{3x^2}=\lim\limits_{x\to\infty}\dfrac{2}{3}=\dfrac{2}{3}$

c. $\lim\limits_{x\to-\infty}\dfrac{3x^2+4x+3}{x^3+x+14}=\lim\limits_{x\to-\infty}\dfrac{3}{x}=0$

Lesson 9: Limits of functions involving indeterminate cases

a) Learning objective:

Evaluate the limits of a functions involving indeterminate cases for both rational and irrational functions

b) Teaching resources

Students' book and other reference textbooks, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

Students will perform well in this lesson if they have good background information on properties and operations of common logarithms and Natural logarithm, domain and range of polynomial, rational , irrational, and logarithmic functions, limits of polynomial, rational and irrational functions.

d) Learning activities

- Invite students to work in groups and do the activity 4.1.8.2 inform the student's book;
- Move around in the class for facilitating students where necessary and ask some challenging questions for better and quality work ;
  - Verify and identify groups with different working steps;
  - Invite groups to present their work.
- Invite all students for a whole class discussion and guide them to guess how to move out the indeterminate cases;
- As a teacher, harmonize the findings from group presentations and use different probing questions, help students to read the examples and content related to the activity 4.1.8.2 in their book for better understanding on how to determine limits of functions involving indeterminate cases.
- After this step, ask students to do the application activity 4.1.8.2. (5 first sub-questions) while the remaining 2 sub-question can be used to evaluate whether lesson objectives were achieved.

Answers for activity 4.1.8.2

Given the function f(x), find a common factor for numerator and denominator and then find the value of f(x) for $x=1;2\ and\ \infty$ on each function. What do you notice for each case?

a) $f(x)=\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{x-1}$, here $f(1)=\frac{0}{0}$ is an indeterminate form $f(2)=3;\quad f(\infty)=\infty$

b) $f(x)=\frac{x^3+x^2-5x-2}{x^2-4}=\frac{x^3-2x^2+3x^2-6x+x-2}{(x-2)(x+2)};x\neq-2,x\neq2$

Which implies that

$$f(x)=\frac{x^2(x-2)+3x(x-2)+(x-2)}{(x-2)(x+2)}=\frac{(x-2)(x^2+3x+1)}{(x-2)(x+2)};$$

therefore; $f(1)=\frac{(1-2)(1^2+3(1)+1)}{(1-2)(1+2)}=\frac{-5}{-3}=\frac{5}{3}$; $f(2)=\frac{0}{0}$ is an indeterminate form; $f(\infty)=\propto$

e) Answers for the application activity 4.1.8.2.

1. The answers for the given question1.

$\lim_{x\to\infty}\frac{3x^3+2x^2-1}{6x^3+x+4}$ the highest degree of numerator is the same as the degree of the denominator, so the answer is a half (½)	$\lim_{x\to-\infty}\frac{(x+3)^2}{x^3+4x^2-8x-4}$ This is equal to 0, the degree of denominator is higher than numerator.	$\lim_{x\to\infty}\frac{4x^3+x^2-1}{x^2-x+4}$ is equal to the infinite as the degree of numerator is higher than the denominator.
$\lim_{x\to-4}\frac{x+1}{x+4}$ This is indeterminate as RHL is different to LHL	$\lim_{x\to3}\frac{x^2+2x+1}{x-3}=\frac{(3)^2+2(3)+1}{3-3}$ $=\frac{16}{0}=\pm\infty$ indeterminate form	$\lim_{x\to\infty}\left(x^2-2x+5\right)$ Indeterminate form , after solving the limit is $+\infty$
$\lim_{x\to-\infty}\left(4x^3+3x^2-6\right)$ The limit is equal to $-\infty$	$\lim_{x\to4}\frac{x^4-16}{x^2-4}$ The answer is 20	$\lim_{x\to4}\frac{\sqrt{x^2-6}-10}{x-4}$ $=\frac{\sqrt{(4)^2-6}-10}{4-4}$ $=\frac{\sqrt{10}-10}{0}=\pm\infty$ Indeterminate form
$\lim_{x\to-\infty}\frac{3x}{\sqrt{9x^2-3x+6}}=\frac{-\infty}{\infty}$ IF, $then$ $\lim_{x\to-\infty}\frac{3x}{\sqrt{9x^2-3x+6}}=\lim_{x\to-\infty}\frac{3x}{-x\sqrt{9-\frac{3}{x}+\frac{6}{x^2}}}=-$		

Lesson 10: Continuity of a function at a point or an interval

a) Learning objective

Apply limits to find out continuity of functions

b) Teaching resources:

Student’s book and other reference textbooks, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they have good background knowledge and skills in previous contents of this unit: properties and operations of common logarithms and Natural logarithm or Naperian logarithm, solving equations and inequalities in the set of real numbers, domain and range of polynomial, rational , irrational, and logarithmic functions, limits of polynomial, rational and irrational functions.

d) Learning activities

- Invite Students to work in group and do the activity 4.2.1 inform the student’s book;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite groups with different working steps to present their work;
- As a teacher, harmonize the findings from Students.
- Help them to explore the content and examples given in the student’s book in relation to the activity 4.2.1 in order to facilitate them to understand better of the application of limits in determination of continuity of a function at a point or on interval
- After the lesson, ask students to do the first question of the application activity 4.2.1 while the second question will be used to evaluate whether lesson objectives were achieved.

Answers for activity 4.2.1

Given the function $f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, x\neq 2\\ 4, \quad \text{x}=2\end{cases}$,

a. $f(2)=4$

b. $\lim_{x\to 2} f(x)=4$

c. $f(2)$ and $\lim_{x\to 2} f(x)$ exist and are equal, $f(2)=\lim_{x\to 2} f(x)=4$

e) Answers of application activity 4.2.1.

1. The function $f(x)=\dfrac{4x+10}{x^2-2x-15}$ is not continuous at $x=-3$ and $x=5$, $f(-3)$ and $f(5)$ are not defined . If one of the three conditions for a function to be continuous at a point is not verified. The function is said to be not continuous at that point.

2. $k=6$, the value of **k** is 6 for which the function is continuous at x = 3.

   This means that $\lim\limits_{x\to3} f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, x\neq 3\\ k, x=3\end{cases}$ exists and equals to $f(3)=6$

Lesson 11: Asymptotes to curve or graph of a function

a) Learning objective:

Extend the concept of limit to determine the asymptotes of the given function

b) Teaching resources:

Student's book and other Reference textbooks, Mathematical set, calculator, Manila paper, graph papers, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

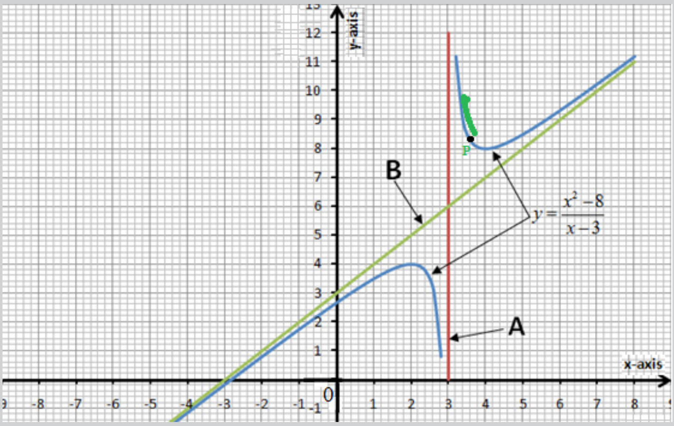
Students will perform well in this lesson if they have good background knowledge and skills on properties and operations of common logarithms and Natural logarithm, domain and range of polynomial, rational , irrational, and logarithmic functions, limits of polynomial, rational and irrational functions.

d) Learning activities

- Invite Students to work in group and do the activity 4.2.2. inform the student's books;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite groups with different working steps to present their work;
- Help them to explore the content and examples given in the student's book in order to better understand the asymptotes of a function to a curve or graph.
- After the lesson, ask students to do the application activity 4.2.2. which can be also used to evaluate whether the lesson objectives were achieved.

Answer for activity 4.2.2

Consider the curve of function $y=\dfrac{x^2-8}{x-3}$,



- As the values of x increases or decreases, the curve comes closer and closer to the line B of the function $f(x)=x+3$.
- As x approaches 3 from the right or from the left, the curve comes closer and closer to the line A of the function $f(x)=3$.

e) Answers for application activity 4.2.2.

Asymptotes for the given functions are calculated and shown below:

1) $f(x)=\dfrac{x^3+x^2-5x-2}{x^3-x^2-2x}$; *Horizontal asymptote*: $\text{HA}\equiv \text{y}=1$

Vertical asymptotes : $VA\equiv x=-1$, $\text{VA}\equiv x=2$, *and* $\text{VA}\equiv x=0$

2) $f(x)=\dfrac{x+3}{x^2+9}$; *Horizontal asymptote* :, *No VA*

3) $f(x)=\dfrac{x^2+3x+1}{4x-9}; VA\equiv x=\dfrac{9}{4}; OA\equiv y=ax+b;$

$$a=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to 2}\frac{x^2+3x+1}{4x^2-9x}=\frac{1}{4}; b=\lim_{x\to\infty}[f(x)-ax]=\lim_{x\to\infty}\left[\frac{x^2+3x+1}{4x-9}-\frac{1}{4}x\right]$$

$$=\frac{4x^2+12x+4-4x^2+9x}{16x-36}=\frac{21}{16};\quad \text{Therefore, } OA\equiv y=\frac{1}{4}x+\frac{21}{16}$$

4) $f(x)=\dfrac{x^2-x-2}{x-2}$; $\text{VA}\equiv x=2$; $OA\equiv y=x+1$

Lesson 12: Continuity and asymptotes of logarithmic functions

a) Learning objective:

Extend the concept of limit to determine the continuity and asymptotes of logarithmic functions

b) Teaching resources:

Student's book and other Reference textbooks, Mathematical set, calculator, Manila paper, graph papers, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they have good background

knowledge and skills in previous contents of this unit: properties and operations of common logarithms and Natural logarithm or Naperian logarithm, solving equations and inequalities in the set of real numbers, domain and range of polynomial, rational, irrational, and logarithmic functions, limits of polynomial, rational and irrational functions.

d) Learning activities

- Invite Students to work in group and do the activity 4.2.3. from the student's books;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite groups with different working steps to present their work;
- Ask students to read the content and examples given in the student's book related to the activity 4.2.3 where they will be able to better understand how to determine the continuity and asymptotes of given logarithmic functions
- After the lesson, guide students to do the application activity 4.2.3. which can be used to evaluate whether lesson objectives were achieved.

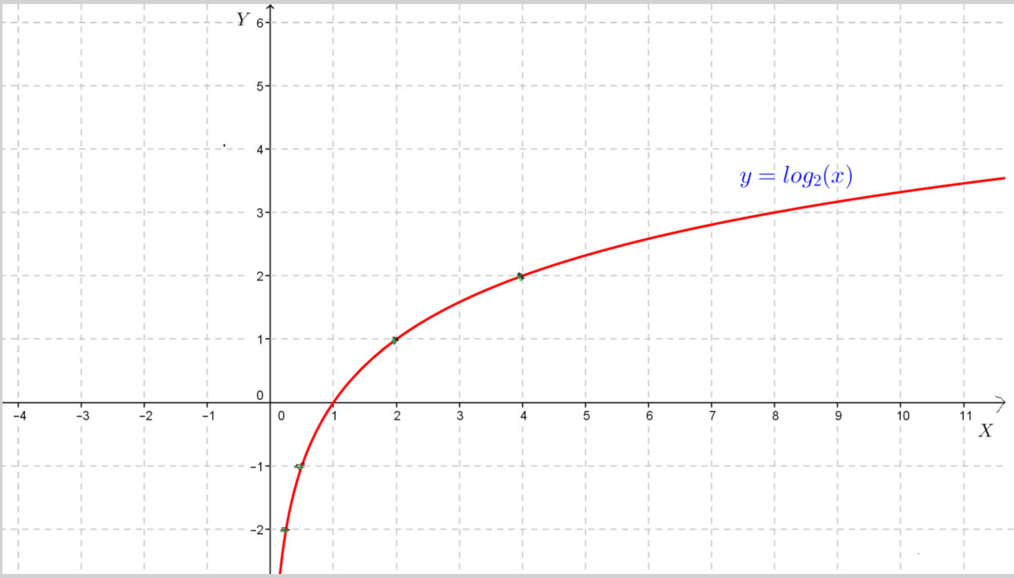
Answer of Activity 4.2.3.

1. Complete the table

$x = x_0$	$y = \log_2 x$	$\lim\limits_{x \to x_0} \log_2 x$
$\frac{1}{4}$	-2	-2
$\frac{1}{2}$	-1	-1
1	0	0
2	1	1
4	2	2

2. For all $x_0 > 0$, $\lim\limits_{x \to x_0} \log_2 x = \log_2 (x_0)$, therefore, $\log_2 x$ is continuous on $]0, +\infty[$

3. The graph of the function $y = \log_2(x)$ can be plotted using a table of values completed using a calculator



For $a > 0$, when $x \to 0, y \to -\infty$, ($\lim\limits_{x \to 0^+} \log_a x$), the line of equation $x = 0$ (the y- axis) is an asymptote to the graph of $f(x) = \log_a x$.

4. The function is continuous on an interval I if

$x_0 \in I, \lim\limits_{x\to x_0^+} f(x) = \lim\limits_{x\to x_0^-} f(x) = f(x_0)$

5. For $f(x)=\ln x$, $\forall x_0 \in]0,+\infty[$, $\lim\limits_{x\to x_0^+} \ln x = \lim\limits_{x\to x_0^-} \ln x = \ln(x_0)$. Therefore,

$f(x)=\ln x$ is continuous on the interval $]0,+\infty[$.

$\lim\limits_{x\to 0^+} \ln x = -\infty$, therefore, the line of equation $x=0$ (the y- axis) is an asymptote to the graph of $f(x)=\ln x$.

II. $\lim\limits_{x\to+\infty} \frac{\ln x}{x} = 0$, $\lim\limits_{x\to 0^+} \frac{\ln x}{x} = -\infty$, $\lim\limits_{x\to 1} \frac{\ln x}{x} = 0$,

$$\lim_{x\to\frac{1}{5}}\left(\frac{\ln x}{x}\right) = \frac{\ln\frac{1}{5}}{\frac{1}{5}} = 5\left(\ln 5^{-1}\right) = -5\ln 5$$

e) Answers of application activity 4.2.3

1) Given the logarithmic function $y=-1+\ln(x+1)$,

   i) The equation of asymptote of the given function $VA \equiv x=-1$, $No\ OA$, $No\ HA$

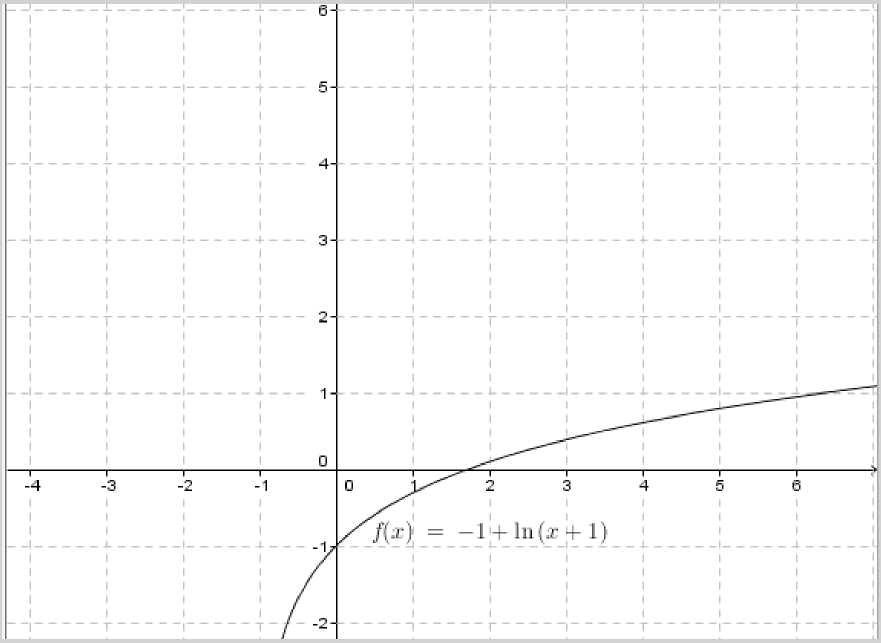
   ii) The domain is $]-1,+\infty[$ and range is $]-\infty,+\infty[$

   iii) The $x-$ intercept is $(e-1,0)$

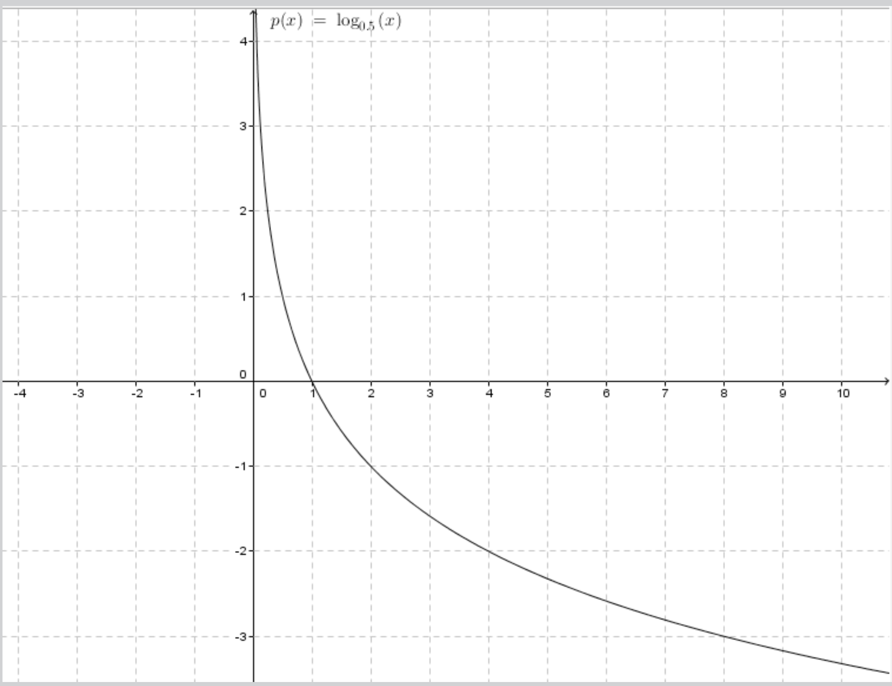
   iv) The $y-$intercept is $(0,-1)$

   v) Another point belonging to the graph is the point (6; 0.95);

vi) The graph of the function $y = -1 + \ln(x+1)$ is as follows:



3. The graph of the logarithmic function of base $a = 0.5$



The function is always positive between 0 and 1, otherwise is negative and it is the opposite of normal logarithmic function whose base is greater than1.

Lesson 13: Continuity and asymptotes of exponential functions

a) Learning objective:

Extend the concept of limit to determine the continuity and asymptotes of the given exponential function.

b) Teaching resources:

Student's book and other Reference textbooks, Mathematical set, calculator, Manila paper, graph papers, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they have good background in previous lessons of unit 3.

d) Learning activities

- Invite Students to work in group and do the activity 4.2.4. from the student's books;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite groups with different working steps to present their work;
- Help them to explore the content and examples given in the student's book where they will be able to evaluate the continuity and asymptotes of given exponential functions
- Ask students to do the application activity 4.2.4. and then do the corrective correction which can help students to make self-evaluation to see whether the lesson objectives were achieved.

Answer of Activity 4.2.4.

Given the function $f(x) = 2^{(x-2)}$

a. $Domain =]-\infty;+\infty[$
$Range =]0;+\infty[$

b $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 2^{x-2} = 2^{-\infty-2} = 0^{+} \Rightarrow H.A \equiv y=0$

c. $f(0) = 2^{0-2} = \frac{1}{4} \Rightarrow y-$ intercept is $\left(0 \quad \frac{1}{4}\right)$

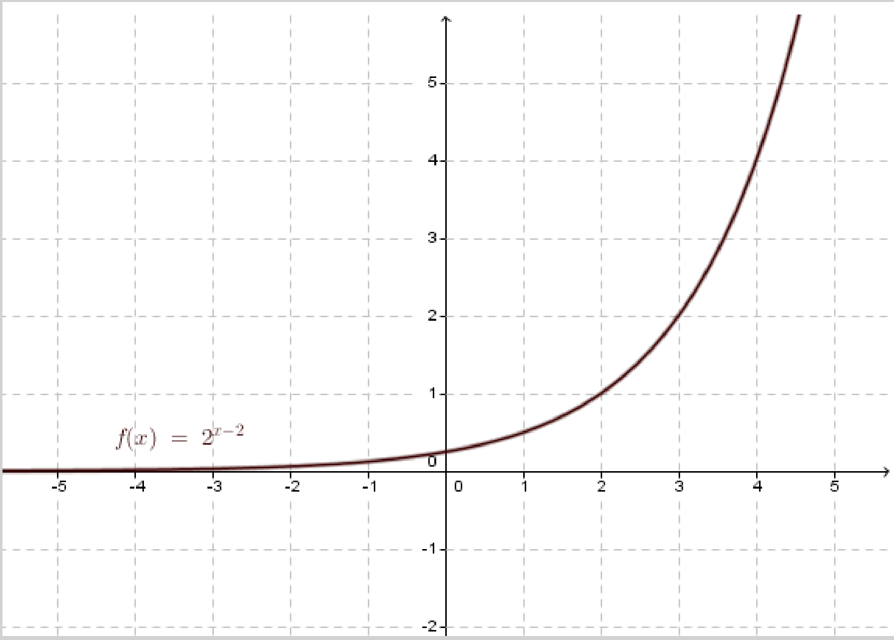
d. $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} 2^{x-2} = 2^{+\infty-2} = +\infty$

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{2^{x-2}}{x} = \frac{+\infty}{+\infty} = \lim_{x\to+\infty} 2^{x-2} = 2^{+\infty-2} = +\infty$$

e. $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} 2^{x-2} = 2^{0^+-2} = 2^{0^+-2} = \frac{1}{4}$

The exponential function $f(x) = 2^{x-2}$ is continuous on its domain.

f. Sketch the graph of $f(x) = 2^{(x-2)}$



e) Answers for the application activity 4.2.4.

a. There is horizontal asymptote given by $y = 3$

b. There is horizontal asymptote given by $y = 2$

c. The horizontal asymptotes given by $HA \equiv \lim\limits_{x\to\pm\infty} \frac{1}{e^x - 1}$,

$HA \equiv \lim\limits_{x\to+\infty} \frac{1}{e^{\infty} - 1} = 0$ and $HA \equiv \lim\limits_{x\to-\infty} \frac{1}{e^{-\infty} - 1} = -1$

d. The asymptote given by $\lim\limits_{x\to-\infty} e^{-1/x^2} = \lim\limits_{x\to-\infty} e^{-1/\infty} = \lim\limits_{x\to-\infty} 1 = 1$

Lesson 14: Application of limit in production, finance and economics related problems

a) Learning objective:

Apply the concept of limits to solve production and problems.

b) Teaching resources:

Student's book and other Reference textbooks, Mathematical set, calculator, Manila paper, graph papers, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they studied well the concept of limit of functions in this unit.

d) Learning activities:

- Invite Students to work in group and do the activity 4.2.5. from the student's books;

- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;

- Invite groups with different working steps to present their work in a whole class discussion;

- Help them to explore the content and examples given in the student's book where they will be able to solve problems by applying the limit of functions.

- Lead students to study the given example and ask them to give more examples from finance where limits of functions are applied.

- Ask students to work in pairs the application activity 4.2.5 from the student's book and then do the corrective correction which can help students to to see whether the lesson objectives were achieved.

e) Answers for activities:

Answer for activity 4.2.5.

$$\lim_{x\to 10} P(x) = \lim_{x\to 10} P_0 e^{rx} = \lim_{x\to 10} 6e^{0.5x}$$

$$= \lim_{x\to 10} 6e^{\frac{x}{2}} = 6e^{\frac{10}{2}} = 6e^{5} = 890.4789546156 \approx 890$$

i.e. The limit $\lim_{x\to 10} P(x)$ is 890

Answer for application activity 4.2.5

Let K be the carrying capacity and, this capacity is given by the following limit.

$$K=\lim_{x\to+\infty}\frac{1}{5.279\times10^{-3}+0.1834e^{-0.031476x}}=\frac{1}{5.279\times10^{-3}+0.1834e^{-0.031476(+\infty)}}$$

$$=\frac{1}{5.279\times10^{-3}+0.1834e^{-\infty}}=\frac{1}{5.279\times10^{-3}+0.1834(0)}$$

$$=\frac{1}{5.279\times10^{-3}+0}=\frac{1}{5.279\times10^{-3}}=189.4$$

Additional information for the teacher

Suppose we have function (logistic function P(x)) describing the size of a population (any number of animals or things that can be produced depending on time at a certain rate), after a given time, x, at a certain fixed rate, r. The highest quantity that can be produced at infinite time is a limit $\lim_{x\to+\infty}P(x)=K$

with $P(x)=\dfrac{K\,P_0\,e^{rx}}{K+P_0\left(e^{rx}-1\right)}$ where the constant $P_0=P(0)$ is the investing capacity (initial quantity).

Other forms of the logistic function

Logistic function $P=P(x)$ is the population size at any given time and it is a solution of a simple differential equation model that can be used to relate the change, $dPdt$, in population P given a (growth) rate r, and a carrying capacity K.

The logistic differential equation can then be expressed as in what follows.

$$\begin{cases}P=Population\\ \dfrac{d}{dx}P=Change\ in\ population\ over\ time\ x\\ \dfrac{K-P}{K}=Percentage\ left\ when\ carrying\ capacity\ K\ is\ reached\end{cases}\Rightarrow\frac{dP}{dx}=rP\frac{K-P}{K}$$

The logistic function $P = P(x)$ is then deduced as a solution of the very simple differential equation $\frac{dP}{dx} = rP\frac{K-P}{K}$ as in what follows.

$$\frac{dP}{dx} = rP\frac{K-P}{K} \Rightarrow \frac{K}{P(K-P)}dP = rdx \Leftrightarrow \left(\frac{1}{P}+\frac{1}{K-P}\right)dP = rdx$$

$$\Rightarrow \int\left(\frac{1}{P}+\frac{1}{K-P}\right)dP = \int r\,dx \Rightarrow \ln|P| - \ln|K-P| = rx + C \Leftrightarrow \ln\left|\frac{K-P}{P}\right| = -rx - C$$

$$\Leftrightarrow \left|\frac{K-P}{P}\right| = e^{-rx-C} \Rightarrow \begin{cases} \frac{K-P}{P} = Ae^{-rx} \\ A = \pm e^{-C} \end{cases} \Leftrightarrow K - P = PAe^{-rx} \Leftrightarrow K = P + PAe^{-rx}$$

$$\Leftrightarrow K = P\left(1 + Ae^{-rx}\right) \Leftrightarrow P = \frac{K}{1+Ae^{-rx}}$$

The constant satisfies the initial condition $\frac{K-P_0}{P_0} = A$ from the initial timing $x = 0$ as in the following statement.

$$x = 0 \Rightarrow A = \frac{K-P_0}{P_0}$$

It follows that the logistic function $P = P(x)$ is given by the quantity $\frac{K}{1+Ae^{-rx}}$ as in the following identity: $P = \frac{K}{1+Ae^{-rx}}$

There are different forms of the logistic function. One of them can also be obtained as in the following statements:

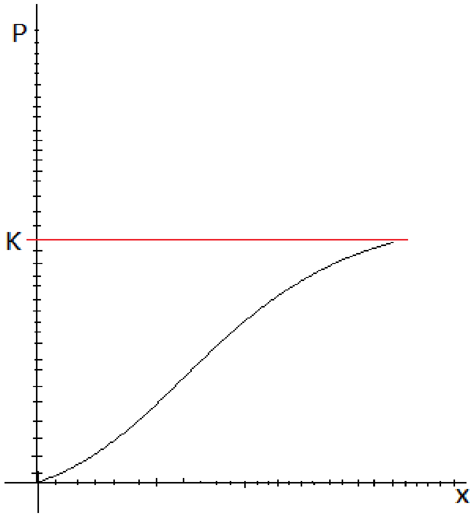
$$P(x) = \frac{KP_0\,e^{rx}}{K+P_0\left(e^{rx}-1\right)} = \frac{\frac{KP_0\,e^{rx}}{P_0\,e^{rx}}}{\frac{K+P_0\left(e^{rx}-1\right)}{P_0\,e^{rx}}} = \frac{K}{\frac{K}{P_0}e^{-rx}+1-e^{-rx}} = \frac{K}{\left(\frac{K}{P_0}-1\right)e^{-rx}+1} = \frac{K}{1+\left(\frac{K-P_0}{P_0}\right)e^{-rx}}$$

$$= \frac{K}{1+Ae^{-rx}} \text{ with } A = \frac{K-P_0}{P_0}$$

i.e $P(x) = \frac{K}{1+Ae^{-rx}}$ with $A = \frac{K-P_0}{P_0}$

A graph of logistic function

The graph of a logistic function can be sketched as in the following diagram.



4.6. Summary of the unit 4

Neighborhood of a real number

Definition: Mathematically, a set N is called a neighborhoods of point P if there exist an open interval I such that $x \in I \subset N$. The collection of all neighborhoods of a point is called the **neighborhood system** at the point.

Condition of existence for a limit

If the limit from the left side is the same as the limit from the right side, say $\lim\limits_{x \to x_0^-} f(x) = \lim\limits_{x \to x_0^+} f(x) = L$, then we write $\lim\limits_{x \to x_0} f(x)$ and we read "the limit of $f(x)$ as x approaches x_0 equals L. Note that $\lim\limits_{x \to x_0} f(x)$ exists if and only if $\lim\limits_{x \to x_0^-} f(x) = \lim\limits_{x \to x_0^+} f(x)$

Properties of limits

Let lim stands for one of the limits $\lim\limits_{x \to a}$, $\lim\limits_{x \to a^-}$, $\lim\limits_{x \to a^+}$, $\lim\limits_{x \to -\infty}$ or $\lim\limits_{x \to +\infty}$. If $\lim f(x)$ and $\lim g(x)$ both exist, say $\lim f(x) = L_1$ and $\lim g(x) = L_2$, then

a) A constant factor can be moved through a limit sign. That is, if k is a constant, then $\lim\left[kf(x)\right] = k \lim f(x)$

b) $\lim\left[f(x) + g(x)\right] = \lim f(x) + \lim g(x) = L_1 + L_2$

c) $\lim\left[f(x) - g(x)\right] = \lim f(x) - \lim g(x) = L_1 - L_2$

d) $\lim\left[f(x) \bullet g(x)\right] = \lim f(x) \bullet \lim g(x) = L_1 \bullet L_2$

e) $\lim\left[\frac{f(x)}{g(x)}\right]=\frac{\lim f(x)}{\lim g(x)}=\frac{L_1}{L_2}$ if $L_2 \neq 0$

If n and m are positive integers, then $\lim\left[f(x)\right]^{\frac{m}{n}}=L_1^{\frac{m}{n}}$ provided that $L_1 \geq 0$ if n is even

Limits are used to determine Asymptotes to curve of a function

There are three types of asymptotes:

1. **Vertical asymptote**

A line with equation $x=x_0$ $(D \equiv x=x_0)$ is called a **vertical asymptote** for the graph of a function $f(x)$ if $\lim\limits_{x\to x_0} f(x)=\pm\infty$ Or $V.A \equiv x=x_0$, *where* $\lim\limits_{x\to x_0} f(x)=\pm\infty$

2. **Horizontal asymptote**

A line with equation $y=L$ $(D \equiv y=L)$ is called a **horizontal asymptote** for the graph of a function $f(x)$ if $\lim\limits_{x\to\pm\infty} f(x)=L$ or $H.A \equiv y=L;$ *where* $\lim\limits_{x\to\pm\infty} f(x)=L$

3. **Oblique asymptote**

If a rational function, $\frac{P(x)}{Q(x)}$, is such that the degree of the numerator exceeds the degree of the denominator, then the graph of $\frac{P(x)}{Q(x)}$ will have an **oblique asymptote (or a slant asymptote)**; that is, an asymptote which is neither vertical nor horizontal.

We perform the division of $P(x)$ by $Q(x)$ to obtain $\frac{P(x)}{Q(x)}=(ax+b)+\frac{R(x)}{Q(x)}$

Where, $ax+b$ is the quotient and $R(x)$ is the remainder.

Another way to find the values of constants a and b is $a=\lim\limits_{x\to\pm\infty}\frac{f(x)}{x}$ $a \neq 0$ and $b=\lim\limits_{x\to\pm\infty}\left[f(x)-ax\right]$

We can write that $O.A \equiv y=ax+b$ where $a=\lim\limits_{x\to\pm\infty}\frac{f(x)}{x}, a \neq 0$ and $b=\lim\limits_{x\to\pm\infty}\left[f(x)-ax\right]$.

4.7. Additional Information for teacher

Given that the knowledge of the teacher must be wider than the one of the learners, the following information is useful for the teacher, though not stated in the learner's book. During teaching and learning process, emphasize on the following:

a) Polynomials are continuous functions

b) If the functions f and g are continuous at c, then

i) $f+g$ is continuous at c

ii) $f-g$ is continuous at c

iii) $f.g$ is continuous at c

iv) $\frac{f}{g}$ is continuous at c if $g(c)\neq 0$, and is discontinuous at c if $g(c)=0$

c) A rational function is continuous everywhere except at the point where the denominator is zero.

d) Piecewise functions (functions that are defined on a sequence of intervals) are continuous if every function is in its interval of definition, and if the functions match their side limits at the points of separation of their intervals.

If $f(x)$ is continuous at $x=a$, then

$\lim_{x\to a} f(x)=f(a)$; $\lim_{x\to a^-} f(x)=f(a)$; $\lim_{x\to a^+} f(x)=f(a)$

and if $f(x)$ is continuous at $x=b$, and $\lim_{x\to a} g(x)=b$, then

$\lim_{x\to a} f(g(x))=f\left(\lim_{x\to a} g(x)\right)=f(b)$

Horizontal asymptote and oblique asymptote do not exist on the same side. That means if $f(x)\to L$ as $x\to+\infty$, there is no oblique asymptote on the right side since there is horizontal asymptote and if $f(x)\to L$ as $x\to-\infty$, there is no oblique asymptote on the left side since there is horizontal asymptote.

4.8. End unit assessment

Answers

1. Given the function $f(x)=\dfrac{x^2+3x+1}{4x-9}$; Find the limits:

   a) $\lim\limits_{x\to\frac{9}{5}} f(x)=\lim\limits_{x\to\frac{9}{5}}\dfrac{x^2+3x+1}{4x-9}=\dfrac{9.64}{-1.8}=-5.3555555556=-\dfrac{241}{45}$;

   b) $\lim\limits_{x\to\pm\infty} f(x)=\pm\infty$,

   c) The function has asymptotes: $VA\equiv x=\dfrac{9}{4}$ and

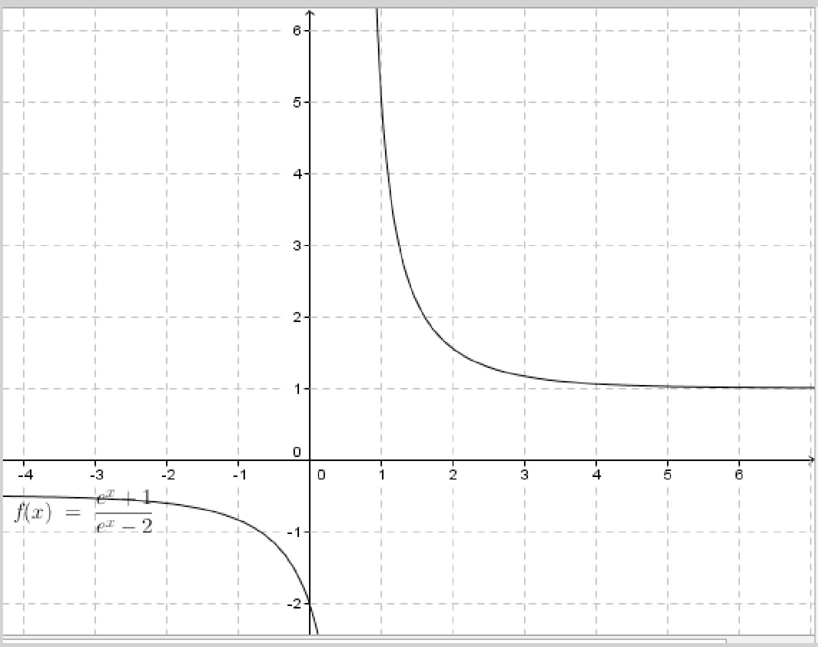
   $OA\equiv ax+b=\dfrac{1}{4}x+\dfrac{21}{16}$

   $a=\lim\limits_{x\to\infty}\dfrac{f(x)}{x}=\dfrac{1}{4}$ *and* $b=\lim\limits_{x\to\infty}(f(x)-ax)=\dfrac{21}{16}$

2. Evaluate the following limit:

$$\lim_{x\to\infty}\frac{e^x+1}{e^x-2}\Leftrightarrow\lim_{x\to-\infty}\frac{e^x+1}{e^x-2}=-\frac{1}{2}\ and\ \lim_{x\to+\infty}\frac{e^x+1}{e^x-2}=\frac{\infty}{\infty}\quad IF$$

and will be solved after studying derivatives. Here use Geogebra to clarify.



3. Find the value for a such that the limit has the indeterminate form of $\frac{0}{0}$

$$\lim_{x\to -2}\frac{3x^2+ax+3}{x^2+x-2}=\frac{3(-2)^2+a(-2)+3}{(-2)^2+(-2)-2}=\frac{12-2a+3}{4-2-2}; \quad when\ a=\frac{15}{2};$$

$$\Rightarrow so, \lim_{x\to -2}\frac{3x^2+\frac{15}{2}x+3}{x^2+x-2}=\frac{0}{0}\ IF$$

4. Given the following functions:

a) $f(x)=\frac{x^2-x-2}{x-2}$ This function is not continuous when $x=2$, otherwise is continuous.

b) $f(x)=\begin{cases}\frac{1}{x^2}, x\neq 0\\ 1, x=0\end{cases}$ This function is not continuous when $x=0$, otherwise is continuous.

c) $f(x)=\begin{cases}x^2-x-2, x\neq 0\\ 1, x=2\end{cases}$ This function exists and is continuous in the set of real numbers.

4.9. Additional activities

You shall consider the levels of your students.

4.9.1. Remedial activity

For the given function $\begin{cases}x+2, x<1\\ x^2, 1\leq x\leq 2\\ 5x-6, x\geq 2\end{cases}$

i) identify the discontinuity,

ii) where $f(x)$ is discontinuous or continuous

Answers

$\lim_{x \to 1^-} f(x) = 3$

$\lim_{x \to 1^+} f(x) = 1$

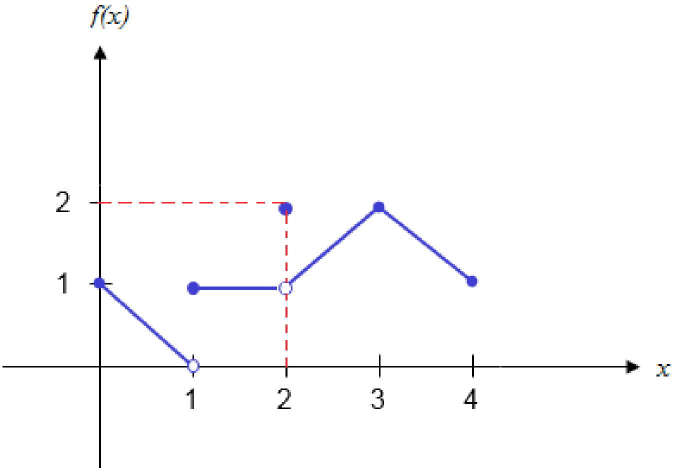
$\lim_{x \to 2^-} f(x) = 4$

$\lim_{x \to 2^+} f(x) = 4$

Conclusion, $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$. Therefore $f(x)$ is discontinuous at $x = 1$; $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$, therefore $f(x)$ is continuous at $x = 2$

4.9.2. Consolidation activity

Observe the graph bellow, analyze it and interpret it then find out if $\lim_{x \to 3} f(x)$ and $\lim_{x \to 2} f(x)$ exist.



Solution:

a) $\lim_{x \to 3^-} f(x) = 2$

$\lim_{x \to 3^+} f(x) = 2$

$f(3) = 2$

Hence, $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$ then $f(x)$ exist.

b) $\lim_{x \to 2^-} f(x) = 1$ **and** $\lim_{x \to 2^+} f(x) = 1$ **but** $f(2) = 2$.

As $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 1$ then $f(x)$ exist.

4.9.3. Extended activity

Evaluate $\lim_{x \to +\infty} \frac{\sqrt{4x^2 - 11x - 3}}{x}$

Solution:

$$\lim_{x\to+\infty} \frac{\sqrt{4x^2 - 11x - 3}}{x} = \frac{\sqrt{\infty - \infty}}{\infty} I.F$$

To evaluate this limit, we try the algebraic manipulations such that the denominator will be cancelled.

$$\lim_{x\to+\infty} \frac{\sqrt{4x^2 - 11x - 3}}{x} = \lim_{x\to+\infty} \frac{\sqrt{4x^2\left(1 - \frac{11}{4x} - \frac{3}{4x^2}\right)}}{x}$$

$$= \lim_{x\to+\infty} \frac{\left(\sqrt{4x^2}\right)\sqrt{1 - \frac{11}{4x} - \frac{3}{4x^2}}}{x}$$

$$= \left(\lim_{x\to+\infty} \frac{\sqrt{4x^2}}{x}\right) \times 1$$

$$= \lim_{x\to+\infty} \frac{\sqrt{4x^2}}{x}$$

$$= \lim_{x\to+\infty} \frac{2\sqrt{x^2}}{x}$$

Recall that $\sqrt{x^2} = \begin{cases} x & if \ x \geq 0 \\ -x & if \ x < 0 \end{cases}$

We need to find the domain of the given function: $Domf = \left]-\infty, -\frac{1}{4}\right] \cup [3, +\infty[$.

As x tends to $+\infty$, $x \in [3, +\infty[$ and then $\sqrt{x^2} = x$.

Thus,

$$\lim_{x\to+\infty} \frac{\sqrt{4x^2 - 11x - 3}}{x} = \lim_{x\to+\infty} \frac{2\sqrt{x^2}}{x}$$

$$= \lim_{x\to+\infty} \frac{2x}{x}$$

$$= 2$$

UNIT 5 FINANCIAL MATHEMATICS

5.1. Key unit competence

Use financial Mathematics techniques in solving production, financial and economical related problems such as simple and compound interests, annuity, sinking funds.

5.2. Prerequisite

Students will perform well in this unit if they are familiar with the contents of the previous units (units 1, 2, 3 and 4) of Senior Four Mathematics for Accounting Profession option, student book and they are skilled in using scientific calculators.

5.3. Cross-cutting issues to be addressed

- Inclusive education (promoting education for all in teaching)
- Peace and value Education (respect the views and thoughts of others during class discussions)
- Financial education (develop sprits of saving, and investments decisions)
- Gender (equal opportunity for boys and girls to participate in class)
- Environment sustainability (compounded, growth, etc)
- Standardized culture (quality of production)

5.4. Guidance on introductory activity

- In groups, facilitate students to read and do introductory activity from the student book;
- Facilitate any discussions, ensure that the work is done without noise and without unnecessary conversations.
- Walk around the classroom to assist students in need;
- Invite group representatives to present their findings and encourage gender in the presentation;
- In the lesson discussion, have students think of different ways to solve the problem.
- Sustain student curiosity about the content of Unit 5 through well-chosen questions.

Answer for introductory activity5.0

1) Use the introductory activity to give an overview of the whole unit, the key concepts that will be discussed, concepts such as Investment, Principal amount P, Interest rate r, Time t or period of investment, Interest I, Total amount accumulated A.

2) i) When you need money in short time, simple interest is better (i.e. less than an year or compounding daily, weekly, monthly, quarterly, semi-annually, …)

ii) No, in long term compounding interest is better. Once you use simple interest you can get less money, compared to when you use compound interest.

Apart from the scenario above, (i) what are your observations when you need to make money in a very short time? (ii) Is it the same if you make money in the long term?

3) Assuming a future amount triples in 5 years, does this relationship indicate an investment opportunity? Is this investment profitable, if so, who can handle it? Why?

5.5. List of lessons

Headings	#	Lesson title/sub-headings	Learning objectives	Number of periods
5.1. Basic concepts of financial Mathematics	0	Introductory activity	To arouse the curiosity of student-teacher on the content of unit 5.	1
	1	Interest and interest rates	Differentiate between the two concepts	1
	2	Simple interest and Compound interest	Apply formula of simple, and compound interests in solving financial related problems	3
	3	Present and Future values	Apply present and future formulae's to solve financial related scenarios	3

5.2. Sequences	1	Introduction to sequences	To arouse the curiosity of students on the introduction to sequences.	1
	2	Arithmetic sequences	Apply arithmetic sequences to solve financial related problems	2
	3	Geometric sequences	Apply geometric sequences to solve financial related problems	2
5.3. Applications of financial Mathematics	1	Annuities	Apply geometric sequences to calculate annuities	3
	2	Mortgage	Use repayment amount to clear the requested loan	2
	3	Sinking funds	Apply discounting rates to clarify the value of investment.	3
	4	Financial Risk management	Apply annuities in calculating premium for both insurer and policy holder	1
End unit assessment				2

Lesson 1: Interest and interest rate

a) Learning objective:

Differentiate between the two concepts, interest and interest rate, and use each of them properly.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on basic operations on numbers, such as addition, subtraction, multiplication and division.

d) Learning activities:

- Invite students to work in groups and do the activity 5.1.1 in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification;
- Verify and identify groups with different usable terminology;
- Invite one member from each group to present their work where they must explain the terminologies provided;
- As a teacher, harmonize the findings from presentation and guide students to define those terms in appropriate ways;
- Use different probing questions and guide students to explore the content and examples given in the student's book and lead them to discover how to define interest and interest rate.
- After this step, guide students to do the application activity 5.1.1 and evaluate whether lesson objectives were achieved.

Answers for activity 5.1.1

a. Ensure that the answer contains the following concepts:

- Interest;
- Interest rate.

b. The interest is the compensation one gets from lending a certain asset;

The interest rate is the interest due, expressed as proportion of the principal, or as a percentage.

e) Answers for Application activity 5.1.1

a) $5\% = \frac{5}{100} = \frac{1}{20}$

b) $\frac{5}{100} \times 1000000 = 50000 FRW$

c) This interest is nominal, because it is the one that is stated.

Lesson 2: Simple and compound interests

a) Learning objective:

Apply formula of simple, and compound interests in solving financial related problems.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils…

c) Prerequisites/Revision/Introduction

Students will learn better this lesson if they have a good background on lesson one of this unit and if they have a good background on basic operations on numbers, such as addition, subtraction, multiplication and division from the previous units.

d) Learning activities:

- Invite students to work in groups and do the activity 5.1.2 in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification to apply formula of simple, and compound interests in solving financial related problems;
- Verify and identify groups with different ways of solving simple and compound interests problems;
- Invite one member from each group to present and explain the steps when applying formula of simple, and compound interests ,in solving financial related problems;
- As a teacher, harmonize the findings from presentation
- Use different probing questions and guide students to explore the content and examples given in the student's book and lead them to discover how to Apply formula of simple, and compound interests in solving financial related problems
- After this step, guide students to do the application activity 5.1.2 and evaluate whether lesson objectives were achieved.

Answers for activity 5.1.2

a) Simple interest:

$$50\ 000\times\frac{8}{100}\times 3=12\ 000 \text{ FRW}$$

Total amount (interest plus principal): $50\ 000FRW+12\ 000FRW=62\ 000FRW$;

b) Compound interest: $50000(1-0.08)^3=62985.6FRW\approx 62986FRW$

c) Compound interest is better than simple interest, in the way that the interest earned is greater than the one for simple interest.

e) Answers for Application activity 5.1.2

1. At bank I: the amount at the end of the year is

$$A=P\left(1+\frac{r}{n}\right)^{nt}=300\ 000\left(1+\frac{0.1}{1}\right)^{1X10}=778\ 122\ FRW$$

At bank II: The amount at the end of the year is:

$$A=Pe^{nt}=300\ 000\times e^{0.098\times 10}=799\ 336FRW$$

You should advise your aunt to invest at bank II because
99 336>778 122 FRW

2. The accumulated amount is: $FV=5\ 000(1+A)^7$ compounded yearly.

3. The annual equivalent rate (AER) is:

   a) For annually, $AER=(1+0.08)-1=0.08$, that is AER=8%

   b) For Semi-annually, $AER=\left(1+\frac{0.08}{2}\right)^2-1=0.08159$, that is the $AER=8.159\%$

   c) For quarterly, $AER=\left(1+\frac{i}{4}\right)^4-1=\left(1+\frac{0.08}{4}\right)^4-1=0.08243$, that is the $AER=8.243\%$

d) For monthly, $AER=\left(1+\frac{i}{12}\right)^{12}-1=\left(1+\frac{0.08}{12}\right)^{12}-1=0.08299$, that is the $AER=8.299\%$

e) For daily, $AER=\left(1+\frac{i}{365}\right)^{365}-1=\left(1+\frac{0.08}{365}\right)^{365}-1=0.08327$, that is the $AER=8.327\%$

4. We have: $900(1+0.04)^t=1000 \Leftrightarrow (1.04)^t=\frac{10}{9}$;

$$t=\frac{\ln\frac{10}{9}}{\ln 1.04}=2.68634\approx 2.69 \text{ years.}$$

5. You will have: $1000(1+\frac{0.02}{4})^4=1020.15$ £

Lesson 3: Present and future values

a) Learning objective:

Apply present and future values in financial and economic decisions making.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the previous units and lessons covered respectively.

d) Learning activities:

- Invite students to work in groups and do the activity 5.1.3 in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification about how to apply formula of Present and future values in solving financial related problems;
- Monitor the work of different groups;

- Invite one member from each group to present their work where they must explain the provided steps to Apply formula of Present and future values in solving financial related problems;
- As a teacher, harmonize the findings from presentation and guide students to define those terms in appropriate ways and apply formula of present and future values in solving financial related problems;
- Use different probing questions and guide students to explore the content and examples given in the student's book and lead them to discover how to apply formula of present and future values in solving financial related problems
- After this step, guide students to do the application activity 5.1.3 and evaluate whether lesson objectives were achieved.

Answers for activity 5.1.3

a) $A < 10\,000$

b) $A:$ Present value;

Future value $ 10000

c) Discounting

e) Answers for Application activity 5.1.3

1. We have: $1500 = 1450(1+\frac{0.04}{12})^{12t} \Leftrightarrow t = \frac{\ln\frac{150}{145}}{12\ln 1.0033} = 0.85751$

year, which is equal to $0.85751 \times 365 days \approx 313 days$

2. Let i_m be the nominal interest rate. Then $(1+i_m)^{12} - 1 = 0.07$;

$(1+i_m)^{12} = 1.07$;

$1+i_m = \sqrt[12]{1.07} = 1.0056$;

$i_m = 0.0565414$, that is 5.654%: annual.

For three months, $\frac{5.654 \times 3}{12} = 1.41\%$

Lesson 4: Introduction to sequences

a) Learning objectives

To define and determine terms of both arithmetic and geometric sequences

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the previous units and lessons covered respectively.

d) Learning activities:

- Invite students to work in groups and do the activity 5.2.1 in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification to define and apply formula of sequences in solving financial related problems;
- Verify and identify groups with different usable steps in Applying formula both arithmetic and geometric sequences in solving financial related problems;
- Invite one member from each group to present their work where they must explain the provided steps to define and apply formula of sequences in solving financial related problems;
- As a teacher, harmonize the findings from presentation and guide students to define those terms in appropriate ways and define and apply formula of sequences in solving financial related problems;
- Use different probing questions and guide students to explore the content and examples given in the student's book and lead them to discover how to apply formula of sequences in solving financial related problems
- After this step, guide students to do the application activity 5.2.1 and evaluate whether lesson objectives were achieved.
- Move around in the class for facilitating students where necessary and give more clarification as they have to give the fraction that represents the part they see when they fold a paper n times;
- Verify and identify groups with different working steps and harmonize their works;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;

- As students, harmonize the findings from presentation and guide students to identify the fraction that represents the part they see when they fold the paper n times;
- Use different probing questions and guide students to explore the content and examples given in the student's book and lead them to discover how to define sequences, series, infinite sequence, and to determine terms of a given sequence.
- After this step, guide students to do the application activity 5.2.1 and evaluate whether lesson objectives were achieved.

Answer for activity 5.2.1

1. Those investments are written as: $T_1, 2T_1, 4T_1, 8T_1, 16T_1, ..., 2^{n-1}T_1$; Given that $T_1 = 126\ 000\ 000\ Frw$, then we have to replace it with their values in the above mentioned investments sequences:

The 1st is T_1 = 126000000 FRW,

The 2nd is $2T_1$ = 252000000 FRW

The 3rd is $4T_1$=504000000FRW

The 4th is $16T_1$ =2016000000FRW ...,

The nth is $2^{n-1}T_1=2^n(63000000)$FRW

2. When they fold once they see $\frac{1}{2}$; When they fold twice they see $\frac{1}{2^2}$;

When they fold 3 times they see $\frac{1}{2^3}$; When they fold n times they see $\frac{1}{2^n}$;… When they fold 10 times they see $\frac{1}{2^{10}}$;…

The list of the fractions obtained is: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \frac{1}{2^{10}}, \ldots \frac{1}{2^n}, \ldots$

e) Answer for Application activity 5.2.1

1) $u_1 = \frac{2\times 1^2}{1^2+1} = 1;\ u_2 = \frac{2\times 2^2}{2^2+1} = \frac{8}{5};\ u_3 = \frac{2\times 3^2}{3^2+1} = \frac{18}{10} = \frac{9}{5}$

2) The five first terms of $\left\{\sqrt{n+1}-\sqrt{n}\right\}_{n=1}^{+\infty}$ are:

$\sqrt{2}-1, \sqrt{3}-\sqrt{2}, 2-\sqrt{3}, \sqrt{5}-2, \sqrt{6}-\sqrt{5}$

3) $\{2n-1\}_{n=1}^{+\infty}$ or $\{2n+1\}_{n=0}^{\infty}$

Lesson 5: Arithmetic sequence

a) Learning objectives

To use the basic concepts of sequences to determine terms of an arithmetic sequence, and find an arithmetic mean of two numbers.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencil…

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to lesson4 of this unit.

d) Learning activities:

- Invite students to work in groups and do the activity 5.2.2 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a teacher, harmonize the findings from presentation and guide students to explain the arithmetic means and identify the common difference of an arithmetic sequence;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to use basic concepts of sequences to determine the general term and different terms of an arithmetic sequence, and find arithmetic means of two terms of an arithmetic sequences.

– After this step, guide students to do the application activity 5.2.2 and evaluate whether lesson objectives were achieved.

Answers for activity 5.2.2

1) a) $\{w_n\}$: 20, 20-2, 20-2.2, 20-2.3, …20-2.n,…0.
This constant is d = -2

b) $\{16-6n\}=16, 10, 4, -2, -6,...\ d=-6$

2) Consider the infinite arithmetic sequence $2,5,8,11,14,....$
a) The first term is $t_1=2$,
b) The common difference is $d=3$
c) The sum is $2+5+8+11+14+17=57$

Observing that $2+17=5+14=8+11$, we have $2+5+8+11+14+17=3(2+17)=57$;

And $3(2+17)=\frac{6(2+17)}{2}$, where 6: number of terms, 2: the first term, and 17: the last term;

Using this pattern, we can project that $t_1+t_2+t_3+...+t_n=\frac{n(t_1+t_n)}{2}$

e) Answers for application activity 5.2.2

1. $-3,-1,1,3,5,7$: the four terms are -3;-1;1 and 5, in that order.
2. $2,5,8,11,14,17,20,23,26,29,32$ The nine numbers are 5;8;11;14;17;20;23;26 and 29, in that order.
3. The arithmetic sequence has $n+2$ terms $t_1,t_2,t_3,...,t_{n+2}$,

$3;a_1,a_2,...,a_8,...a_{n-2},a_{n-1},a_n;54$,

where $t_1=3;t_2=a_1;t_3=a_2;...;t_9=a_8;...;t_{n-1}=a_{n-2};...$

Let d be the common difference. Then, from $t_{n+2}=t_1+(n+1)d$, we have $54=3+(n+1)d$ Solving for nd, we have: $nd=51-d$ (1)

But also,

$$\frac{a_8}{a_{n-2}}=\frac{3}{5}\Leftrightarrow 5a_8=3a_{n-2}$$

$$\Leftrightarrow 5(3+8d)=3[3+(n-2)d]$$

$$\Leftrightarrow 3nd=6+46d \qquad (2)$$

Solving simultaneously (1) and (2): $\begin{cases} nd = 51 - d \\ 3nd = 6 + 46d \end{cases}$; $d = 3$ and $n = 16$.

4. Form an arithmetic progression of three numbers such that the sum of its terms is 30 and the sum of the squares of its terms is 332**.**

Solution: Let the second term be x. The first term is $x-d$ and the third term is $x+d$ where d is the common difference. Now, $x-d+x+x+d=30 \Rightarrow 3x=30$ or $x=10$

Also, $(x-d)^2 + x^2 + (x+d)^2 = 332$

Or $(10-d)^2 + 100 + (10+d)^2 = 332$

Or $2d^2 = 32 \Rightarrow d = \pm 4$

Therefore, the progression is 6, 10, 14 or 14, 10, 6

5. The sum S_n of the first n terms of an arithmetic sequence with first term $t_1 = 6$, common difference $d = 4$ and nth term $t_n = t_1 + (n-1)d = 8 + (n-1)4 = 4 + 4n$, the sum is

$$S_n = \frac{n(t_1 + t_n)}{2} = \frac{n(8+4+4n)}{2} = 2n(n+3)$$

6. The sum S_n of the first n terms of an arithmetic sequence with first term $t_1 = 5$, common difference $d = 4$ and nth term $t_n = t_1 + (n-1)d = 5 + (n-1)4 = 1 + 4n$, the sum is

$$S_n = \frac{n(t_1 + t_n)}{2} = \frac{n(5+1+4n)}{2} = n(2n+3)$$

7. From $S_n = \frac{n(t_1 + t_n)}{2} = \frac{n[t_1 + t_1 + (n-1)d]}{2} = \frac{n(7n-5)}{2} = 396$, solving the equation $n^2 - 5n - 792 = 0$, we find $n = 11$ (the positive value). Therefore, the sequence contains 11 terms

8. The bottom row requires 100 tiles and the top row, 50 tiles. Since each successive row requires one less tile, the total number of tiles required is $S = 100 + 98 + 96 + ... + (100 - (n-1)) + ... + 50$.

As $100-(n-1)=50$, we have: $100-n+1=50$ Which gives $n=51$.

This is the sum of an arithmetic sequence; the common difference is -1. The number of terms to be added is $n=51$ with the first term $u_1=100$ and the last term $u_n=50$. The sum S is $S=\frac{n}{2}(u_1+u_n)=\frac{51}{2}(100+50)=3825$. In all, 3825 tiles will be required.

Lesson 6: Geometric sequences

a) Learning objectives

To use basic concepts and formulas of sequences to determine terms, to calculate the sum of the first "n" terms, and find geometric means of two numbers.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites

Students will learn better in this lesson if they refer to all previous lessons especially this Unit (from the lesson 1 to lesson 5) and all previous Units respectively..

d) Learning activities:

- Invite students to work in groups and do the activity 5.2.3 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarifications on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work;
- As a teacher, harmonize the findings from presentation and guide students to make a geometric sequence of numbers, its general term and the common ratio; also, she/he explain geometric means of two terms in a geometric sequence;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to determine terms of a geometric sequence.

– Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to find geometric mean of two numbers.

– After this step, guide students to do the application activity 5.2.3 and evaluate whether lesson objectives were achieved.

Answer for activity 5.2.3

We can display the amounts the students received in a table as follows:

Place	Money gotten
1st	$u_1 = 100\ 000 Frw$
2nd	$u_2 = \frac{1}{2}(100\ 000 Frw) = 50\ 000 Frw$
3rd	$u_3 = \frac{1}{2}(50\ 000 Frw) = 25\ 000 Frw$
4th	$u_4 = \frac{1}{2}(25\ 000 Frw) = 12\ 500 Frw$
5th	$u_5 = \frac{1}{2}(12\ 500 Frw) = 6\ 250 Frw$

The total of their money is

$$u_1 + u_2 + u_3 + u_4 + u_5$$

$$= 100\ 000 + \frac{1}{2}(100000) + \frac{1}{2^2}(100000) + \frac{1}{2^3}(100000) + \frac{1}{2^4}(100000)$$

The money for the first is 100 000 Frw, this is far greater than the money for the fifth student which is 6 250 Frw. When you win at the first place, the money will be greater than the one of the next winner.

The money for the student who passed at the n^{th} place is $u_n = \frac{1}{2^{(n-1)}} 100\ 000 Frw$

To determine the total amount of money for n students , we calculate the sum from $u_1 = 100\ 000 Frw$ to $u_n = \frac{1}{2^{(n-1)}} 100\ 000 Frw$ and we find

$$S_n = 100000 \times \frac{\left(1 - \left(\frac{1}{2}\right)^n\right)}{\left(1 - \frac{1}{2}\right)}$$

e) Answer for application activity 5.2.3

1. We have: $u_1 = \frac{1}{4}; u_7 = \frac{1}{256}$

From $u_7 = q^6 u_1, q^6 = \frac{u_7}{u_1} = \frac{\frac{1}{256}}{\frac{1}{4}} = \frac{1}{64}; q = \sqrt[6]{\frac{1}{64}} = \frac{1}{2}$; the geometric sequence is:

$$\frac{1}{4};\frac{1}{8};\frac{1}{16};\frac{1}{32};\frac{1}{64};\frac{1}{128};\frac{1}{256}$$

2. In the same way, we obtain: $2;\frac{2}{3};\frac{2}{9};\frac{2}{27};\frac{2}{81};\frac{2}{243};\frac{2}{729}$

3. a) Find the geometric mean between 2 and 98 is $\sqrt{2\times 98} = 14$

   b) The geometric mean between $\frac{3}{2}$ and $\frac{27}{2}$ is $\sqrt{\frac{3}{2}\times\frac{27}{2}} = \frac{9}{2}$

4. The next two terms are $2,916$ and $26,244$

5. Let x and y be the two numbers. Then $\frac{x+y}{2} = 34$ and $\sqrt{xy} = 16$ .Solving for x and y, we find: $x = 64; y = 4$ or $x = 4; y = 64$

6. The sum of the first 8 terms is $S_8 = 32\left[\frac{1-(-\frac{1}{2})^8}{1-(-\frac{1}{2})}\right] = \frac{2^{14}-2^6}{3} = \frac{85}{4}$

In fact, $32+(-16)+8+(-4)+2+(-1)+\frac{1}{2}+(-\frac{1}{4}) = \frac{85}{4}$

7. The sum of the first n terms is $S_n = 0.99\left[\frac{1-(0.99)^n}{1-0.99}\right]$

8. Since $S_n = \frac{5^n-4^n}{4^{n-1}} = (1)\left[\frac{1-(\frac{5}{4})^n}{1-\frac{5}{4}}\right]$, the first term is 1 and the common ratio is $\frac{5}{4}$

Lesson 7: Annuities

a) Learning objectives

To apply the financial Mathematics, concepts and formulas of sequences to solve financial and economic related problems.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils…

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons especially geometric sequences.

d) Learning activities:

- Invite students to work in groups and do the activity 5.3.1 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarifications on eventual challenges they may face during their work; Verify and identify groups with different working problems;
- Invite one member from each group to present their work;
- As a teacher, harmonize the findings from presentation and guide students to apply geometric sequence in handling annuities scenarios, and provide the required explanations.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to apply the annuities in solving financial and economical related scenarios.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to find the annuities and its applicability.
- After this step, guide students to do the application activity 5.3.1 and evaluate whether lesson objectives were achieved.

Answer for activity 5.3.1

	The payment is made at the		Payment frequency and Compounding frequency are	
	End of the period	Beginning of the period	equal	Different
a)	Yes	No	Yes	No
b)	Yes	No	No	Yes
c)	No	Yes	No	Yes

e) Answer for the application activity 5.3.1

1. $P\left(1+\frac{0.03}{12}\right)^{12t}=2P$ .Solving for t, we find $t=23.1$ years

2. $P\left(1+\frac{0.09}{12}\right)^{12\times 6}=12\ 000$;

$$P=\frac{12\ 000}{\left(1+\frac{0.09}{12}\right)^{72}}\approx 7\ 007;$$

3. $P(1+0.0845)^{t}=3P$;

$$t=\frac{\ln 3}{\ln 1.0845}=13.54 \text{ years}$$

Lesson 8. Mortgage

a) Learning objectives

Apply arithmetic and geometric sequences to solve mortgage problems

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites

Students will learn better in this lesson if they refer to lesson7 and geometric sequences.

d) Learning activities:

- Invite student-teachers to work in groups and give them instructions on how they can do the introductory activity 1.0 found in unit 1 of student's book;
- Guide students to read and analyse the questions insisting on the analysis of the given data and to determine the number of insects that will be there in second, third, fourth,…n^{th} generation.
- Invite some group members to present groups' findings, then try to harmonize their answers; try to insist on the list formed by the number of insects at any generation and the generalisation (number of insects at n^{th} generation).
- Basing on students' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answer for activity 5.3.2

a) A **mortgage** is a loan

b) Huge amount of money, to be repaid in regular periodic installments, generally for a long period of time, with a compounded interest on the amount not yet paid.

e) Answer for application activity 5.3.2

Substituting for variables, the periodic payment is P=200 000 FRW, the annual rate r=10%=0.01, the number of payments per year n=12(since the payment is monthly), the number of years to cover the mortgage is t=20. Replacing each quantity by its value in the formula , we have

$$P=\frac{\frac{rM}{n}}{1-(1+\frac{r}{n})^{-nt}}$$ .Then:

$$200000=\frac{\frac{0.1M}{12}}{1-(1+\frac{0.1}{12})^{-12(20)}}$$

Solving for M, we find M=20 808 156.36 FRW; the mortgage is 20 808 156 FRW

Lesson 9. Sinking funds

a) Learning objectives

Apply arithmetic and geometric sequences to solve sinking fund problems

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils…

c) Prerequisites

Students will learn better in this lesson if they refer to lesson7 and geometric sequences.

d) Learning activities:

- Invite students to work in groups and give them instructions on how they can do activity 5.3.3
- Request students to read and analyse the questions
- Invite some group members to present groups' findings, then try to harmonize their answers
- Basing on students' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this lesson.

Answer for activity 5.3.3

In problem (a) I am saving, and the money I invest is less than the amount I aim at obtaining.

In problem (b) I am paying a debt and the money owed is greater than the money received

e) Answer for application activity 5.3.3

This is an example of a sinking fund. The payment P required four times a year to accumulate 1 800 US Dollars in 2 years (8 payments at a rate of interest of $i = \frac{0.06}{4} = 0.015$ per payment period) obeys:

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right]$$

$$1,800 = P\left[\frac{\left((1+0.015)^8 - 1\right)}{0.015}\right]$$

$P = 213.45$ US Dollars

Your quarterly payment will be $P = 213.45$ US Dollars

Lesson10. Financial Risk Management

a) Learning objectives

Apply arithmetic and geometric sequences to solve sinking fund problems

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites

Students will learn better in this lesson if they refer to lesson7 and geometric sequences.

d) Learning activities:

- Invite students to work in groups and give them instructions on how they can do activity 5.3.3
- Request students to read and analyse the questions
- Invite some group members to present groups' findings, then try to harmonize their answers
- Basing on students' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be learnt in this lesson.

Answer for activity 5.3.4

All the four scenarios are about financial risks, these are the situations that implies the investor to not perform fully his/her duties

e) Answer for application activity 5.3.4

The uncle's request may not be granted by the bank, since it is a risk for the following reasons:

- 20 years time may find the uncle already retired from work, thus unable to pay the loan
- The uncle's monthly income may not allow him to pay the loan

5.6. Summary of the unit 5

1. Interest (I), interest rate(i in percentage), principal(P), total amount(A), after a period of time(n years) are related by;

   For simple interest: $I = n \times i \times P$ and $A = P + I$

   For compound interest: $A = (1+i)^n P$; $I = A - P$

2. A compound interest can be discrete or continuous.

   For discrete compound interest (daily, weekly, monthly, semiannually, annually):

   $A = P\left(1+\frac{r}{n}\right)^{nt}$, where P:principal, $i = \frac{r}{n}$ and n:the number of time , in a year, the interest is compounded, t:the total number of years.

   For continuous compound interest: $A = Pe^{rt}$

3. An interest rate can be nominal or effective:

   Nominal interest rate: the stated interest rate

   Effective interest rate:
   AER : the annual equivalent rate, APR: the annual percentage rate

   For discrete compound interest: $AER = \left(1+\frac{i}{m}\right)^m - 1$,

   where i:nominal interest rate, m: compounding periods

   For continuous compound interest: $AER = e^r - 1$

4. The values of the same investment at different times are not the same, thus yielding to the concepts of Present value(PV) and future value(FV):

   $PV = \frac{FV}{(1+i)^t}$, where **FV =** the **future value** of the investment at the end of t years; **t =** the number of years in the future; **i =** the interest rate or annual interest rate and **PV =** the current or **present value** of the sum of money that you intend to invest today. The process of finding **PV** from **FV** is called **discounting.**

5. Let (t_n) be a sequence, with first term t_1, common difference d (for arithmetic sequence) or common ratio r (for geometric sequence).We have:

	Arithmetic sequence	Geometric sequence
nth term	$t_n = t_1 + (n-1)d$	$t_n = t_1 r^{n-1}$
Common difference Or common ratio	$t_{n+1} - t_n = d$ (constant independent of n)	$\frac{t_{n+1}}{t_n} = r$ (constant independent of n)
Sum of the first n terms	$S_n = \frac{n(t_1 + t_n)}{2}$	$S_n = t_1\left(\frac{1-r^n}{1-r}\right)$

6. An **annuity** is a sequence of continuous payments of equal amounts, at fixed intervals, for a fixed period

For an **Ordinary annuity:** the payment is made at the end of a period interval.

The formula for calculating the present value and the future value are given by:

$$PV = C\left[\frac{1-(1+i)^{-n}}{i}\right];\ FV = C\left[\frac{(1+i)^n - 1}{i}\right];$$

For an **Annuity due:** the payment is made at the beginning of a period interval.

The formula for calculating the present value and the future value are given by:

$PV = C\left[\frac{1-(1+i)^{-n}}{i}\right](1+i)$; $FV = C\left[\frac{(1+i)^n - 1}{i}\right](1+i)$,where, C: cash flow: the constant amount deposited once at an interval time; i: interest rate; n: number of payments;

FV: future value; PV: present value;

A **deferred annuity** is an Annuity in which payback does not start until a specified time in future: you can split the period into two: period before the start of the payback and the period from the start of the payback

7. A **mortgage** is a loan, generally of huge amount of money, to be repaid in regular periodic installments, generally for a long period of time, with a compounded interest on the amount not yet paid.

The formula for mortgage payment is $P = \frac{M \times i \times (1+i)^n}{(1+i)^n - 1}$ or $P = \frac{\frac{rM}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}$

8. **Sinking fund:** is a special type of annuity; it consists in saving periodically a constant amount with a compound interest, to meet a future financial need. The formula for future value of an ordinary annuity applies for sinking funds

$$FV = C\left[\frac{(1+i)^n - 1}{i}\right]$$

9. Investors need to be aware of financial risks .It is, therefore, crucial to avoid risks, if avoidable, or manage properly a financial risk, for smooth running of a financial activity.

5.7. Additional information

Here the teachers have to emphasize on the applications of sequences and financial mathematics in general by solving problems related to production, financial and economical related situation in real life. These include how to estimate the population growth, to calculate the interest rate problems, to calculate the mortgage payment, to determine any other application in finance and economics related problems, etc.

The teacher should do his/her best to not improvise exercises, since they have to be realistic and updated.

5.8. End unit assessment

Answers:

1. a) $i_e=\left(1+\frac{i}{m}\right)^m-1=(1.03)^2-1=0.061$

   Therefore, the EAR is 6.1%

   b) $i_e=e^r-1=e^{0.06}-1=0.068$

   Therefore, the EAR is 6.8%

2. By calculating the present values:

   For project A: $400000(1.1)^{-6}=225789.572; 225789.572>200000$

   For project B: $480000(1.1)^{-5}=298042.235; 298042.235<300000$

   Therefore, project A is the best.

3. We have:

   $$100000+100000(1.12)+100000(1.12)^2+100000(1.12)^3+...+100000(1.12)^{11}$$

   Sum of the terms of a geometric sequence with first term 100000 and common ratio 1.12;

   $$S_{12}=100000\left(\frac{1-1.12^{12}}{1-1.12}\right)=2413316.667$$

4. The mortgage is $M=\frac{P\times(\frac{r}{n})}{1-(1+\frac{r}{n})^{-nt}}=\frac{6000000\times0.0075}{1-(1+0.0075)^{-300}}=50351.782$

   The monthly amount to pay is $\approx 50352 Frw$

5. The total amount is

   $$100+2(100)+2^2(100)+2^3(100)+...+2^{29}(100)=100(2^{29}-1)$$

   This amount is too big, the father cannot honor his promise.

5.9. Additional activities

5.9.1. Remedial activities

1. Find the 20th term of the following arithmetic progressions and calculate the sum of first 20 terms

   a. 2, 6, 10, 14, …

   b. -5, -3.5, -2, -0.5, …

Solution:

a. $u_{20} = 78, S_{20} = 800$

b. $u_{20} = 23.5, S_{20} = 185$

2. In an arithmetic progression, the sum of the 8th and 14th terms is 50. The 5th term is equal to 13. Find that progression.

Solution:

$$5,7,9,11,13,15,17,19,21,23,25,27,29,31,....$$

3. If Linda deposits $1300 in a bank at 7% interest compounded annually, how much will be in the bank 17 years later?

Solution:

$$FV = PV\left(1+i\right)^{n} = 1300\left(1+0.07\right)^{17} = \$4106.46$$

5.9.2. Consolidation activities

1. Find x consecutive integers given that the first number is 8 and their sum is x^3.

Solution:

$$8,9,10$$

2. In a geometric progression, we have

   a. $u_1 = 3, r = 4, n = 5$; find u_n and sum of terms.

   b. $u_n = \frac{3}{64}, u_1 = 12, n = 9$; find r and sum of terms

Solution:

a. $u_5 = 3(4)^4 = 768$; $S_5 = 3\left(\frac{1-4^5}{1-4}\right) = 4^5 - 1 = 1023$

b. $12r^8 = \frac{3}{64} \Leftrightarrow r = \sqrt[8]{\frac{1}{256}} = \frac{1}{2}$;

$$S_9 = 12\left[\frac{1-\left(\frac{1}{2}\right)^9}{1-\frac{1}{2}}\right] = 23.9531$$

3. How long will it take our money to triple in a bank account with an annual interest rate of 8.45% compounded annually?

Solution

Let $PV = x$ .Then $x(1+0.0845)^n = 3x \Leftrightarrow 1.0845^n = 3$;

$n = \frac{\ln 3}{\ln 1.0845} = 13.54320$; It requires 13.54 years

5.9.3. Extended activity

How much money would you need to deposit today at 9% annual interest compounded monthly to have \$12000 in the account after 6 years?

Solution:

$$PV = \frac{12000}{(1+0.075)^{72}} = 65.73$$

You need to invest \$ 65.73

REFERENCES

1. Arem, C. (2006). Systems and Matrices. In C. A. DeMeulemeester, Systems and Matrices (pp. 567-630). Demana: Brooks/Cole Publishing 1993 & Addison-Wesley1994.

2. Goemans, M. X. (2015, March 17). google.com . Retrieved March 2, 2022, from cs.cmu.edu: *http://ww.cs.cmu.edu>user>Goemans-LP-notes*

3. John J. Schiller, M. S. (2009, 2012). Schaum's Outline of Probability and Statistics (4th ed.). New York, United States of America , New York: McGraw Hill.

4. Joseph G. & Rosenstein, e. a. (2006). Discrete Mathematics in the schools. In N. C. al., Discrete Mathematics through Applications (Vol. 1, pp. 699-790). Demana: NCTM and AMS.

5. Markov. (2006). Matrix Algebra and Applications. In Matrix Algebra and Applications (pp. p173-208.).

6. Michael Sullivan, e. a. (© 2011, 2008, 2006). Finite Mathematics an applied approach (eleventh edition ed.). Chicago State University: JOHN WILEY & SONS, INC.

7. NCERT, p. (2021). google. Retrieved March 04, 2022, from google.com : *https://ncert.nic.in>textbook>pdf>kest108*

8. Polya, G. (2006). A New Aspect of Mathematical Method. In E. G. I. M. Gelfand, & George (Ed.), Functions and Graphs (Second Edition ed., Vol. 1, pp. 69-168). Demana: Dale Seymour.

9. REB, R.E (2020). Mathematics for TTCs Year 2 Social Studies Education. Student's book

10. REB, R.E (2020). Mathematics for TTCs Year 3 Social Studies Education. Student's book

11. REB, R.E (2020). Mathematics for TTCs Year 1 Science Mathematics Education. Student. 's book

12. REB, R.E (2020). Mathematics for TTCs Year 1 Social Studies Education. Student's book

13. Rossar, M. (1993, 2003). Basic Mathematics For Economists . London and New York: Ttaylor&Francis Group.

14. Shirley, J. K. (2002). Calaculus: A modern approach . Fall .

15. Stirzaker, D. (2003). Elementary Probability. Cambridge, United States of America: Cambridge University Press.

16. william G. McCallum & Deborah Hughes-Hallett, e. a. (2009). Applied calculus (4th edition ed.). (e. a. David Dietz & Shannon Corliss, Ed.) R.R. Donnelley / Jefferson: Wiley .

17. Wrede R., M. D. (2010). Schaum's Outline Advanced Calculus (3rd ed.). (2. 1. Copyright 2010, Ed.) New York Chicago San Francisco LisbonLondon Madrid Mexico City Milan New Delhi, San Jose State University,Rensselaer Polytechnic InstituteHartford Graduate Center, San Juan Seoul Singapore Sydney Toronto: McGraw-Hill.

18. ZACH. (2018). Introduction to Simple Linear Regression . © 2021 Statology.